Quantum Mechanical Search and Harmonic Perturbation

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Perturbation theory in quantum mechanics studies how quantum systems interact with their environmental perturbations. Harmonic perturbation is a rare special case of time-dependent perturbations in which exact analysis exists. Some important technology advances, such as masers, lasers, nuclear magnetic resonance, etc., originated from it. Here we add quantum computation to this list with a theoretical demonstration. Based on harmonic perturbation, a quantum mechanical algorithm is devised to search the ground state of a given Hamiltonian. The intrinsic complexity of the algorithm is continuous and parametric in both time T and energy E. More precisely, the probability of locating a search target of a Hamiltonian in N-dimensional vector space is shown to be $1/(1 + cNE^{-2}T^{-2})$ for some constant c. This result is optimal. As harmonic perturbation provides a different computation mechanism, the algorithm may suggest new directions in realizing quantum computers.

Quantum physics can in principle speed up solving the unsorted-database search problem with a quadratic improvement over classical algorithms, as was first demonstrated by Grover [1]. This problem was originally formulated as to identify a target item in the fewest queries to a black-box database. An important reformulation by Farhi et al. [2, 3] phrased the problem as to search the target state with some special eigenvalue of a given Hamiltonian $\mathcal{H}$, which represents the database. In particular, constant and adiabatic perturbations were proposed in [2] and [3], respectively, for quantum search. These methods yield the same quadratic speed-up as Grover's construct [4, 5]. Moreover, the adiabatic computation is equivalent to standard quantum computation (in terms of unitary transformations) [6]. Although some physical implementations have been demonstrated in realizing quantum search algorithms, e.g. [7–9], they may not be scalable to solve large problem instances without some fundamental breakthroughs [10]. Searching alternative computation models may suggest new ways of building quantum computers.

Despite the success of the constant and adiabatic perturbations in quantum search, the applicability of perturbations based on fast time-varying Hamiltonians remains an open problem. This letter exploits harmonic perturbation for quantum computation. A new computation model is proposed, inspired by the well-studied harmonic perturbation of two-state systems in quantum mechanics. By preparing a system in one or the other of its two states initially, such perturbation induces an absorption-emission cycle (the phenomenon of periodic oscillation of the probability for the system being found at one of its two states) at the resonance condition. More specifically, consider a two-state physical system of Hamiltonian $\mathcal{H} = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2|$, with $E_2 > E_1$, in a sinusoidal potential $\mathcal{V}(t) = \gamma e^{i\omega t}|1\rangle\langle 2| + \gamma e^{-i\omega t}|2\rangle\langle 1|$. The state evolution of the system is governed by the Schrödinger equation

$$i\hbar \frac{\partial \psi(t)}{\partial t} = \widetilde{\mathcal{H}}(t)\psi(t) \qquad (1)$$

with $\psi(t) = c_1(t)|1\rangle + c_2(t)|2\rangle$ and $\widetilde{\mathcal{H}}(t) = \mathcal{H} + \mathcal{V}(t)$. For the initial condition $c_1(0) = 0$ and $c_2(0) = 1$, the respective probabilities of finding the system in states $|1\rangle$ and $|2\rangle$ are of exact solutions

$$|c_1(t)|^2 = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega - \omega_{21})^2/4} \sin^2(\Omega t) \qquad (2)$$

$$|c_2(t)|^2 = 1 - |c_1(t)|^2 \qquad (3)$$

where $\omega_{21} \equiv (E_2 - E_1)/\hbar$ and $\Omega \equiv \sqrt{\frac{\gamma^2}{\hbar^2} + \frac{(\omega - \omega_{21})^2}{4}}$. At resonance, $\omega = \omega_{21}$, the probability of finding the system in the ground state $|1\rangle$ oscillates with period $\pi\hbar/\gamma$, and reaches 1 at time $(2k + 1)\pi\hbar/(2\gamma)$, $k = 0, 1, 2, \ldots$. This phenomenon reveals the potential usefulness of harmonic perturbation in searching the ground state of a given Hamiltonian even with arbitrarily multiple states.

Before proceeding, we formulate the database search problem as follows. Given a Hamiltonian $\mathcal{H} = \Sigma_{j=1}^{N} E_j|j\rangle\langle j|$, we are asked to find state $|g\rangle$ such that E_g is the minimum among E_j's, i.e., $|g\rangle$ is the ground state. In the sequel, we assume any state $|j\rangle$ is a configuration of n binary digits; thus, $N = 2^n$. Moreover, we assume that the state of the underlying physical system is measurable such that its n-bit configuration (e.g., spin orientations of spin-1/2 particles measured along some axis) is completely determined, and that its corresponding energy under $\mathcal{H}$ can be observed thereafter. Unless otherwise stated, we shall, for simplicity, focus on the energy distribution of Grover's search problem. That is, one out of the N states is the ground state of energy $E^{\perp}$, and the other $N - 1$ states are excited states of energy

Algorithm 1: Naïve Ground-State Search
input: a Hamiltonian $\mathcal{H}$ and its ground-state energy $E^\perp$
output: the ground state of $\mathcal{H}$
begin
01 measure state, yielding say $|g_1\rangle$.
02 **if** $E_{g_1} = E^\perp$ **return** $|g_1\rangle$
03 apply harmonic perturbation $\mathcal{V}_{[g_1,\gamma,\omega]}(t)$ with $\omega = \Delta/\hbar$.
04 measure state at time $\pi\hbar/(2\gamma)$, yielding say $|g_2\rangle$.
05 **return** $|g_2\rangle$
end

FIG. 1: The procedure of a naïve quantum search.

$E^\top = E^\perp + \Delta$ with energy gap $\Delta \geq 0$. In our algorithm, we consider that $E^\perp$ is known and Δ is adjustable.

In principle, if the perturbation potential is designed properly, knowing the initial (excited) state and energy gap of a given N-state system, one can apply the resonance frequency to induce an oscillatory transition almost solely between the initial state and the ground state, similar to the two-state case. Measuring the system at the right time achieves the highest probability of locating the ground state. How this peak probability is related to N is our main concern. To gain an insight on the optimality limit of quantum search using harmonic perturbation, we begin with a simple trial (in Figure 1) and then proceed with an optimized procedure (in Figure 2).

Figure 1 sketches a (non-optimal) quantum search procedure. The algorithm starts with a measurement in the state basis to enforce the underlying physical system collapsing to some state, say $|g_1\rangle$. If the corresponding eigenenergy of the input Hamiltonian $\mathcal{H}$ in state $|g_1\rangle$ equals the ground state energy $E^\perp$, the algorithm has found the target and returns $|g_1\rangle$ immediately at Step 2. Otherwise, harmonic perturbation is applied using the sinusoidal potential $\mathcal{V}_{[j,\gamma,\omega]}(t)$ with

$$\langle p|\mathcal{V}_{[j,\gamma,\omega]}(t)|q\rangle = \begin{cases} \gamma e^{i\omega t} & \text{if } q = j \text{ and } q \neq p \\ \gamma e^{-i\omega t} & \text{if } p = j \text{ and } p \neq q \\ 0 & \text{otherwise} \end{cases}$$

for indices $p, q = 1, \ldots, N$. By replacing index j with g_1 and letting $\omega = (E_{g_1} - E^\perp)/\hbar$, the perturbation $\mathcal{V}_{[g_1,\gamma,\omega]}(t)$ at Step 3 induces an oscillatory probability for the system swinging mainly between state $|g_1\rangle$ and the unknown ground state. The algorithm measures, at Step 4, the state of the system at time $\pi\hbar/(2\gamma)$, when the system situates in the ground state with the highest probability.

We analyze the condition under which this peak probability is independent of the effect of N and is close to 1. For $N = 2$, the returned state $|g_2\rangle$ of the algorithm is the ground state with certain. However, it is not the case for $N > 2$. To see why, we solve the Schrödinger equation of the N-state system with Hamiltonian $\widetilde{\mathcal{H}}(t) = \mathcal{H} + \mathcal{V}_{[g_1,\gamma,\omega]}(t)$. To simplify the discussion, assume without loss of generality that $|1\rangle$ is the

Algorithm 2: Optimized Ground-State Search
input: a Hamiltonian $\mathcal{H}$ and its ground-state energy $E^\perp$
output: the ground state of $\mathcal{H}$
begin
01 measure state, yielding say $|g_1\rangle$.
02 **if** $E_{g_1} = E^\perp$ **return** $|g_1\rangle$
03 apply harmonic perturbation $\mathcal{V}^o_{[g_1,\gamma,\omega]}(t)$ with $\omega = \Delta/\hbar$.
04 measure state at time $\pi\hbar/(2\gamma)$, yielding say $|g_2\rangle$.
05 **if** $E_{g_2} = E^\perp$ **return** $|g_2\rangle$
06 apply harmonic perturbation $\mathcal{V}^e_{[g_2,\gamma,\omega]}(t)$ with $\omega = \Delta/\hbar$.
07 measure state at time $\pi\hbar/(2\gamma)$, yielding say $|g_3\rangle$.
08 **return** $|g_3\rangle$
end

FIG. 2: The procedure of an optimized quantum search.

ground state and $|g_1\rangle = |2\rangle$ is the initial state. Let $c_k(t)$ denote the probability amplitude of state $|k\rangle$ at time t. Then the original N first-order differential equations from Equation (1) can be reduced to three due to the equivalence of $c_3(t), c_4(t), \cdots, c_N(t)$. Hence, at resonance $\omega = \Delta/\hbar \equiv \omega_R$, the reduced equations in terms of $b_k(t) \equiv e^{iE_k t/\hbar} c_k(t)$ are

$$i\hbar\dot{b}_1(t) = \gamma b_2(t)$$
$$i\hbar\dot{b}_2(t) = \gamma b_1(t) + (N-2)\gamma e^{-i\omega_R t} b_3(t)$$
$$i\hbar\dot{b}_3(t) = \gamma e^{i\omega_R t} b_2(t)$$

To solve $|b_1(t)|^2$ and thus $|c_1(t)|^2$, apply Laplace transform $\mathcal{L}$ on these equations and solve for $B_1(s) \equiv \mathcal{L}\{b_1(t)\}$ with initial conditions $b_2(0) = 1$ and $b_k(0) = 0$ for $k \neq 2$. We derive

$$B_1(s) = \frac{-i\bar{\gamma}}{(s^2 + \bar{\gamma}^2)} \frac{1}{\left(1 + \frac{\bar{\gamma}^2}{(s^2+\bar{\gamma}^2)}\Lambda_1\right)} \tag{4}$$

where $\bar{\gamma} \equiv \gamma/\hbar$ and $\Lambda_1 \equiv (N-2)s/(s + i\omega_R)$. From the inverse Laplace transform of $B_1(s)$, the exact solution of $c_1(t)$ can be derived. For $N = 2$ and thus $\Lambda_1 = 0$, $c_1(t)$ reduces to Equation (2). For general $N > 2$, we omit listing the sophisticated expression of $c_1(t)$ as the trial construct is not our final destination. However, an analysis shows that, for a fixed constant $\bar{\gamma}$ (among other possibilities), to maintain a constant peak probability, $\Pr = \max_t |c_1(t)|^2$, we need $\omega_R \propto N$. Thus, the algorithm has time complexity $O(1)$ due to the fixed perturbation amplitude γ, and has energy complexity $O(N)$ due to $\omega_R \in O(N)$. This constant complexity in time and linear complexity in energy can be explained in the s-domain by observing that N and ω_R in Λ_1 of Equation (4) are of a first-order relation. That is, by maintaining $N/\omega_R = \epsilon$ for some small constant ϵ, the effect of Λ_1 is negligible and $|c_1(t)|^2 \approx \sin^2(\bar{\gamma}t)$. Hence, the algorithm achieves the same linear resource complexity as the classical algorithms for database search.

Based on the insight from the s-domain analysis, we obtain an optimized algorithm. Figure 2 sketches a refined quantum search algorithm similar to that of Fig-

ure 1. It differs from the procedure of Figure 1 mainly in an additional iteration, and in the applied perturbations $\mathcal{V}^o_{[j,\gamma,\omega]}(t)$ and $\mathcal{V}^e_{[j,\gamma,\omega]}(t)$, where

$$\langle p|\mathcal{V}^o_{[j,\gamma,\omega]}(t)|q\rangle = \begin{cases} \gamma e^{i\omega t} & \text{if } (p = j \neq q \text{ and } q \text{ even}) \text{ or} \\ & (q = j \neq p \text{ and } p \text{ odd}) \\ \gamma e^{-i\omega t} & \text{if } (q = j \neq p \text{ and } p \text{ even}) \text{ or} \\ & (p = j \neq q \text{ and } q \text{ odd}) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\langle p|\mathcal{V}^e_{[j,\gamma,\omega]}(t)|q\rangle = \begin{cases} \gamma e^{i\omega t} & \text{if } (p = j \neq q \text{ and } q \text{ odd}) \text{ or} \\ & (q = j \neq p \text{ and } p \text{ even}) \\ \gamma e^{-i\omega t} & \text{if } (q = j \neq p \text{ and } p \text{ odd}) \text{ or} \\ & (p = j \neq q \text{ and } q \text{ even}) \\ 0 & \text{otherwise} \end{cases}$$

for indices $p, q = 1, \ldots, N$. Thus $\mathcal{V}^o_{[j,\gamma,\omega]}(t)$ and $\mathcal{V}^e_{[j,\gamma,\omega]}(t)$ are conjugate to each other. For instance, $\mathcal{V}^o_{[2,\gamma,\omega]}(t)$ in a 4×4 matrix reads

$$\begin{pmatrix} 0 & \gamma e^{i\omega t} & 0 & 0 \\ \gamma e^{-i\omega t} & 0 & \gamma e^{-i\omega t} & \gamma e^{i\omega t} \\ 0 & \gamma e^{i\omega t} & 0 & 0 \\ 0 & \gamma e^{-i\omega t} & 0 & 0 \end{pmatrix}.$$

Because $\mathcal{V}^o_{[j,\gamma,\omega]}(t)$ and $\mathcal{V}^e_{[j,\gamma,\omega]}(t)$ induce noticeable probability oscillation only when the ground state, say $|g\rangle$, situates at an odd (i.e. g odd) and even (i.e. g even) position, respectively, Algorithm 2 requires one more perturbation-and-measurement iteration than Algorithm 1. If the ground state, not returned in Step 2 of Figure 2, situates at an odd (respectively even) position, it will be returned at Step 5 (respectively Step 8) almost for sure. The algorithm can be repeated to further enhance its correctness probability, which we omit for simplicity. (Notice that the initialization measurements at Steps 1 and 4 determine the parameter j of $\mathcal{V}^o_{[j,\gamma,\omega]}(t)$ and $\mathcal{V}^e_{[j,\gamma,\omega]}(t)$. Alternatively one may prepare the initial state in some specific state, and thus j is a fixed constant rather than a varying parameter depending on the measurements.)

We analyze the condition under which the probability for the algorithm finding the search target is independent of the effect of N and is close to 1. To simplify our discussion, again we assume without loss of generality that $|1\rangle$ is the ground state of $\mathcal{H}$ and the first measurement in Step 1 of Figure 2 yields $|g_1\rangle = |2\rangle$. To compute the probability that the ground state $|1\rangle$ is correctly returned in Step 5, we solve the Schrödinger equation of an N-state system with Hamiltonian $\widetilde{\mathcal{H}}(t) = \mathcal{H} + \mathcal{V}^o_{[2,\gamma,\omega]}(t)$. Let $c_k(t)$ be the probability amplitude of state $|k\rangle$ at time t. Then the original N first-order differential equations can be reduced to four due to the equivalence of odd coefficients $c_3(t), \ldots, c_{N-1}(t)$ and the equivalence of even coefficients $c_4(t), \ldots, c_N(t)$. Hence, at resonance $\omega = \Delta/\hbar \equiv \omega_R$, the

reduced equations in terms of $b_k(t) \equiv e^{iE_k t/\hbar} c_k(t)$ are

$$i\hbar \dot{b}_1(t) = \gamma b_2(t)$$
$$i\hbar \dot{b}_2(t) = \gamma b_1(t) + \frac{(N-2)}{2}\gamma(e^{-i\omega_R t}b_3(t) + e^{i\omega_R t}b_4(t))$$
$$i\hbar \dot{b}_3(t) = \gamma e^{i\omega_R t}b_2(t)$$
$$i\hbar \dot{b}_4(t) = \gamma e^{-i\omega_R t}b_2(t)$$

To solve $|b_1(t)|^2$ and thus $|c_1(t)|^2$, we apply Laplace transform $\mathcal{L}$ on these equations and solve for $B_1(s) \equiv \mathcal{L}\{b_1(t)\}$ with initial conditions $b_2(0) = 1$ and $b_k(0) = 0$, $k \neq 2$. We derive

$$B_1(s) = \frac{-i\bar{\gamma}}{(s^2 + \bar{\gamma}^2)} \frac{1}{(1 + \frac{\bar{\gamma}^2}{(s^2+\bar{\gamma}^2)}\Lambda_2)} \tag{5}$$

where $\Lambda_2 \equiv (N-2)s^2/(s^2 + \omega_R^2)$. Taking inverse Laplace transform $\mathcal{L}^{-1}\{B_1(s)\}$ and assuming $N \gg 1$, we get

$$b_1(t) \approx \frac{-i\omega_R}{\sqrt{N\bar{\gamma}^2 + \omega_R^2}} \sin\left(\frac{\bar{\gamma}\omega_R}{\sqrt{N\bar{\gamma}^2 + \omega_R^2}}t\right) \tag{6}$$

By Equation (6), the peak probability Pr of finding the search target equals $\max_t |c_1(t)|^2 = 1/(1 + N\bar{\gamma}^2\omega_R^{-2})$; the period of the probability oscillation of $|c_1(t)|^2$ is $\tau = \pi\sqrt{1 + N\bar{\gamma}^2\omega_R^{-2}}/\bar{\gamma}$. Since on average the search target can be found by running the algorithm $1/\text{Pr}$ times each of which takes time τ, the total time complexity is of $\tau/\text{Pr} \in O((1 + N\bar{\gamma}^2\omega_R^{-2})^{\frac{3}{2}}/\bar{\gamma})$. On the other hand, the energy complexity is of $O(\omega_R)$. It can be verified that maintaining a constant Pr achieves the tightest upper bound of resource complexity. As a result, when Pr is maintained as a constant (as we shall assume in the sequel), $\bar{\gamma}$ is the only parameter affecting time complexity T. Letting E denote the energy complexity, we can write $\text{Pr} = 1/(1 + cNE^{-2}T^{-2})$ for some constant c. In essence, the time-energy product complexity is of $O(\sqrt{N})$.

The foregoing analysis assumes that the ground state is in an odd position. Suppose that the ground state is rather in some even position. Then $\mathcal{V}^o_{[g_1,\gamma,\omega]}(t)$ of Step 3 does not induce a state oscillation (and thus $|g_2\rangle$ is the same as $|g_1\rangle$ with high probability) while $\mathcal{V}^e_{[g_2,\gamma,\omega]}(t)$ of Step 6 does. In this case, a similar analysis holds to derive the probability that the algorithm returns a correct answer at Step 8.

To understand the improvement of Algorithm 2 over Algorithm 1, compare Equations (4) and (5). We see that the undesirable effect of N in Λ_2 is nullified by ω_R^2 while that in Λ_1 is nullified by ω_R. As a consequence, for constant γ, Algorithms 1 and 2 are of energy complexities $O(N)$ and $O(\sqrt{N})$, respectively. Ideally, if we can construct a Λ such that the effect of N is cancelled out by ω_R^d for a larger exponent d, then Pr can be maintained as a constant with a lower resource complexity. Unfortunately, it is to be shown that d is at most 2, that is, no improvement is possible by introducing more different phases and/or amplitudes to the non-zero entries

of $\mathcal{V}^o_{[j,\gamma,\omega]}(t)$ and $\mathcal{V}^e_{[j,\gamma,\omega]}(t)$. (Note that replacing the zero entries of $\mathcal{V}^o_{[j,\gamma,\omega]}(t)$ and $\mathcal{V}^e_{[j,\gamma,\omega]}(t)$ with non-zero elements introduces undesirable amplitude leakage to states other than the initial and target ground states. Hence we only need to consider modifying the non-zero entries.)

Consider a general perturbation potential (a Hermitian matrix) with non-vanishing entries only in the row and column indexed by some initial state $|j\rangle$ similar to $\mathcal{V}_{[j,\gamma,\omega]}(t)$. These entries can have arbitrary amplitudes and frequencies. (However, one of the frequencies must equal ω_R such that resonance is possible.) Assuming without loss of generality $|1\rangle$ and $|2\rangle$ to be the ground and initial state, respectively, we solve the corresponding Schrödinger equations and can write

$$B_1(s) = \frac{-i\bar{\gamma}}{(s^2 + \bar{\gamma}^2)} \; \frac{1}{(1 + \frac{\bar{\gamma}^2}{(s^2+\bar{\gamma}^2)}\Lambda)}$$

with $\hbar\bar{\gamma}$ being the amplitudes for entries $|1\rangle\langle2|$ and $|2\rangle\langle1|$, and

$$\Lambda \equiv s\left(\frac{\alpha_1}{s + i\omega_1} + \frac{\alpha_2}{s + i\omega_2} + \cdots + \frac{\alpha_m}{s + i\omega_m}\right) \quad (7)$$

where index m is polynomial in n (thus $m \ll N = 2^n$), α_j's are positive real numbers, ω_j's are of the form $a_j\omega_R^{x_j}$ for real constants a_j and x_j, and $\omega_j \neq \omega_k$ for $j \neq k$. (Note that α_j is obtained from the product of some complex number and its complex conjugate, and thus is positive.)

Theorem 1 *Let d equal the largest exponent in terms of ω_R in the denominator of Equation (7) minus that in the nominator. Then, $d \leq 2$ for all possible assignments to the constant parameters α_j, a_j, and x_j of Λ.*

Since $d = 2$ is achieved, the quantum search algorithm of Figure 2 is optimal.

We noted very recently that the work [11] presented an algorithm on quantum search with resonance, which is in fact no better than the classical algorithm. There a perturbation the same as the potential $\mathcal{V}_{[j,\gamma,\omega]}(t)$ in our trial construct was proposed. The misconceived quantum improvement in the paper was due to the ignorance of time-energy product complexity and to the over-approximated analysis [12]. Moreover, the formulation assumed that the eigenenergy of the search target is unique and known *a priori*. However, under this assumption, the energy eigenstates could be pre-computed. Thus the target eigenstate can be known once the eigenenergy is specified without even resort to quantum search. In contrast, we assume that the excited energy eigenstates are degenerate and there is another observable commute with the Hamiltonian that further determines the states.

To summarize, based on an exact analysis this letter answered affirmatively and constructively the open question whether fast time-varying Hamiltonians can be exploited in quantum search. The presented procedure using harmonic perturbation is optimal and achieves a quadratic speed-up over classical algorithms similar to other methods [1–3]. Under this new computation model, it may suggest different approaches to the realization of quantum computers.

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[1] L. Grover. A fast quantum mechanical algorithm for database search. In *Proc. the 28th ACM Symposium on the Theory of Computing*, pages 212–219, 1996.

[2] E. Farhi and S. Gutmann. Analog analogue of a digital quantum computation. *Physical Review A*, 57(4):2403–2406, 1998.

[3] E. Farhi, J. Goldstone, S. Gutmann, and M. Sipser. Quantum computation by adiabatic evolution. quant-ph/0001106, 2000.

[4] W. van Dam, M. Mosca, and U. Vazirani. How powerful is adiabatic quantum computation? In *Proc. the 42th IEEE Symposium on Foundations of Computer Science*, pages 279–287, 2001.

[5] J. Roland and N. Cerf. Quantum search by local adiabatic evolution. *Physical Review A.*, 65(4):042308(6), 2002.

[6] D. Aharonov, W. van Dam, J. Kempe, Z. Landau, S. Lloyd, and O. Regev. Adiabatic quantum computation is equivalent to standard quantum computation. In *Proc. the 45th IEEE Symposium on Foundations of Computer Science*, 2004.

[7] L. Vandersypen, M. Steffen, M. Sherwood, C. Yannoni, G. Breyta, and I. Chuang. Implementation of a three-quantum-bit search algorithm. *Applied Physics Letters*, 76(5):646–648, 2000.

[8] X. Peng, X. Zhu, X. Fang, M. Feng, M. Liu, and K. Gao. Experimental implementation of Hogg's algorithm on a three-quantum-bit NMR quantum computer. *Physical Review A*, 65:042315, 2002.

[9] M. Steffen, W. van Dam, T. Hogg, G. Breyta, and I. Chuang. Experimental implementation of an adiabatic quantum optimization algorithm. *Physical Review Letters*, 90(6):067903(4), 2003.

[10] J. Preskill. Quantum computing: pro and con. *Proc. R. Soc. Lond. A*, 454:469–486, 1998.

[11] A. Romanelli, A. Auyuanet, and R. Donangelo. Quantum search with resonances. *Physica A.*, 360:274–284, 2006.

[12] The underlying Hamiltonians for numerical simulations in [11] were taken from the quantum harmonic oscillator and the two-dimensional quantum rotor, which have eigenenergies $E_m = E_0(m + 1/2)$ and $E_m = E_0 m^2$, respectively. Since these two Hamiltonians are non-degenerate, the highest (or average) eigenenergies in consideration must be no less than $\Omega(N)$ and $\Omega(N^2)$, respectively, which corresponds to our notion of energy complexity. On the other hand, the Schrödinger equations to be solved were oversimplified and resulted in a problematic conclusion about the complexity improvements. In particular, since energy gaps $(E_i - E_j) \gg 1/\sqrt{N}$ for any $i \neq j$, the probability amplitude $c_k(t)$ of any state other than the initial and target states was assumed to be 0. However, this assumption can be invalid because the effect of $N - 2$ such small amplitudes may not be ignored.

Appendix

Theorem 1   *Proof:*

Expanding Λ of Equation (7), we have

$$\Lambda = \frac{1}{(s + i\omega_1)\cdots(s + i\omega_m)}\{s^m + \cdots +$$
$$(\sum_j \alpha_j \sum_{k \neq j}(\prod_{l \neq j, l \neq k} \omega_l))s^2 + (\sum_j \alpha_j \prod_{k \neq j} \omega_k)s\}. \quad (8)$$

We first show that, if $d > 0$, the largest exponent in terms of ω_R in the denominator of Equation (8) must result from the product of all ω_j's. Let $\sigma = x_1 + \cdots + x_m$; then the product term $\omega_1\omega_2\cdots\omega_m$ equals $a\omega_R^\sigma$ for some constant $a \neq 0$. For contradiction, suppose that the product term $\omega_1\omega_2\cdots\omega_m$ is not the only highest order term in ω_R. Then there must exist at least one $\omega_j = a_j\omega_R^{x_j}$ with $x_j \leq 0$. Let $a'\omega_R^{\sigma'}$ for $a' \neq 0$ be the product term of all ω_j's with $x_j > 0$. Because $a'\omega_R^{\sigma'}$ must appear somewhere in both the denominator and nominator of Equation (8), d must equal 0. Hence, to have $d > 0$, all ω_j's must have $x_j > 0$. That is, the product term $\omega_1\omega_2\cdots\omega_m$ contributes to the largest exponent of ω_R in the denominator.

To have $d > 2$, the exponent of ω_R in the nominator of Equation (8) cannot be too large. Observe that, for the resonance condition to hold in harmonic perturbation, there must exist some $x_k = 1$. Moreover, there are more than one such x_k's. Otherwise, the product term $\omega_1\cdots\omega_{k-1}\omega_{k+1}\cdots\omega_m$ cannot be cancelled out. Its existence in the nominator of Equation 8 results in $d \leq 1$, which violates the desired condition. Because of these constraints and $x_j > 0$ for $j = 1, \ldots, m$, the coefficients

of s and s^2 in the nominator of Equation (8) must equal zero. That is,

$$\sum_j \alpha_j \prod_{k \neq j} \omega_k = 0, \text{ and} \quad (9)$$

$$\sum_j \alpha_j \sum_{k \neq j}\left(\prod_{l \neq j, l \neq k} \omega_l\right) = 0 \quad (10)$$

must be satisfied. Multiplying (9) by $(\frac{1}{\omega_1} + \cdots + \frac{1}{\omega_m})$ yields

$$\sum_j \alpha_j \frac{\prod_{k \neq j} \omega_k}{\omega_j} + \sum_j \alpha_j \sum_{k \neq j}\left(\prod_{l \neq j, l \neq k} \omega_l\right) = 0. \quad (11)$$

By recognizing that the second term of Equation (11) equals 0 by Equation (10), it is immediate that

$$\sum_j \alpha_j \frac{\prod_{k \neq j} \omega_k}{\omega_j} = 0. \quad (12)$$

Again, multiplying Equation (12) by the product term $\omega_1\omega_2\cdots\omega_m$ yields

$$\sum_j \alpha_j \prod_{k \neq j} \omega_k^2 = 0. \quad (13)$$

Since α_j's are positive real numbers, the left-hand side of Equation (13) must be greater than zero unless all ω_j's equal zero, which violates the condition that $\omega_j \neq \omega_k$ for $j \neq k$. Thus, Equation (13) does not hold, and $d \not> 2$. Moreover, since $d = 2$ is achieved by Algorithm 2, the theorem follows. ∎