Henkin Quantifiers and Boolean Formulae

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Abstract. Henkin quantifiers, when applied on Boolean formulae, yielding the so-called dependency quantified Boolean formulae (DQBF), offer succinct descriptive power specifying variable dependencies. Despite their natural applications to games with incomplete information, logic synthesis with constrained input dependencies, etc., DQBF remain a relatively unexplored subject however. This paper investigates their basic properties, including formula negation and complement, formula expansion, and prenex and non-prenex form conversions. In particular, the proposed DQBF formulation is established from a synthesis perspective concerned with Skolem-function models and Herbrand-function countermodels.

1 Introduction

Henkin quantifiers [9], also known as branching quantifiers among other names, generalize the standard quantification by admitting explicit specification, for an existentially quantified variable, about its dependence on universally quantified variables. In addition to mathematical logic, Henkin quantifiers appear not uncommonly in various contexts, such as natural languages [12], computation [2], game theory [11], and even system design. They permit the expression of (in)dependence in language, logic and computation, the modelling of incomplete information in noncooperative games, and the specification of partial dependencies among components in system design, which is the main motivation of this work.

When Henkin quantifiers are imposed on first-order logic (FOL) formulae, it results in the formulation of independence-friendly (IF) logic [10], which was shown to be more expressive than first-order logic and exhibit expressive power same as existential second-order logic. However one notable limitation among others of IF logic under the game theoretical semantics is the violation of the law of the excluded middle, which states either a proposition or its negation is true. Therefore negating a formula can be problematic in terms of truth and falsity. In a game theoretical viewpoint, it corresponds to undetermined games, where there are cases under which no player has a winning strategy. Moreover, the winning strategies of the semantics games do not exactly correspond to Skolem and Herbrand functions in synthesis applications although syntactic rules for negating IF logic formulae were suggested in [7, 6].

When Henkin quantifiers are imposed on Boolean formulae, it results in the so-called dependency quantified Boolean formulae (DQBF), whose satisfiability lies in the complexity class of NEXPTIME-complete [11]. In contrast to QBF, which is PSPACE-complete, DQBF offers more succinct descriptive power than QBF provided that NEXPTIME is not in PSPACE. By expansion on universally quantified variables, a DQBF can be converted to a QBF with the cost of exponential blow up in formula size [4, 5].

This paper studies DQBF in a synthesis perspective. By distinguishing formula negation and complement, the connections between Skolem and Herbrand functions are established. While the law of the excluded middle holds for negation, it does not hold for complement. The special subset of the DQBF whose truth and falsity coincide with the existence of Skolem and Herbrand functions, respectively, is characterized. Our formulation provides a unified view on DQBF models and countermodels, which encompasses QBF as a special case. Some fundamental properties of DQBF are studied in Section 3, and the potential application of DQBF on Boolean relation determinization for input constrained function extraction is discussed in Section 4. Discussions and conclusions are then given in Section 5 and Section 6, respectively.

2 Preliminaries

As conventional notation, a set is denoted with an upper-case letter, e.g., V; its elements are in lower-case letters, e.g., $v_i \in V$. The ordered version (i.e., vector) of $V = \{v_1, \ldots, v_n\}$ is denoted as $\boldsymbol{v} = (v_1, \ldots, v_n)$. Two vectors $\boldsymbol{v}$ and $\boldsymbol{v}'$ satisfy $\boldsymbol{v}' \subseteq \boldsymbol{v}$ if $V' \subseteq V$. Substituting a term t (respectively a vector of terms $\boldsymbol{t} = (t_1, \ldots, t_n)$) for some variable v (respectively a vector of variables $\boldsymbol{v} = (v_1, \ldots, v_n)$) in a formula ϕ is denoted as $\phi[v/t]$ (respectively $\phi[\boldsymbol{v}/\boldsymbol{t}]$ or $\phi[v_1/t_1, \ldots, v_n/t_n]$). A formula ϕ under some truth assignment α to its variables is denoted as $\phi|_\alpha$.

2.1 Quantified Boolean Formulae

A *quantified Boolean formula* (QBF) Φ over variables $V = \{v_1, \ldots, v_k\}$ in the *prenex form* is expressed as

$$Q_1 v_1 \cdots Q_k v_k.\phi,$$

where $Q_1 v_1 \cdots Q_k v_k$, with $Q_i \in \{\exists, \forall\}$, is called the *prefix*, denoted Φ_{pfx} ,and ϕ, a quantifier-free formula in terms of variables V, is called the *matrix*, denoted Φ_{mtx}. We call variable v_i in a QBF an *existential variable* if $Q_i = \exists$, or a *universal variable* if $Q_i = \forall$. A QBF is of *non-prenex form* if its quantifiers are scattered around the formula without a clean separation between the prefix and the matrix. Unless otherwise said, we shall assume that a QBF is in the prenex form and is totally quantified, i.e., with no free variables. As a notational convention, unless otherwise specified we shall let $X = \{x_1, \ldots, x_n\}$ be the set of universal variables and $Y = \{y_1, \ldots, y_m\}$ existential variables.

Given a QBF Φ over variables V, the *quantification level* $\ell : V \to \mathbb{N}$ of variable $v_i \in V$ is defined to be the number of quantifier alternations between $\exists$ and $\forall$ from the outermost variable to variable v_i in Φ_{pfx}, e.g., $\ell(v_1) = \ell(v_2) = 0$, $\ell(v_3) = 1$, and $\ell(v_4) = 2$ for QBF $\exists v_1 \exists v_2 \forall v_3 \exists v_4.\phi$.

Any QBF Φ over variables $X \cup Y$ can be converted into the well-known *Skolem normal form* [13]. In the conversion, every appearance of $y_i \in Y$ in Φ_{mtx} is replaced by its respective newly introduced function symbol F_{y_i} corresponding to the *Skolem function* of y_i, which refers only to the universal variables $x_j \in X$ with $\ell(x_j) < \ell(y_i)$. These function symbols are then existentially quantified before (on the left of) other universal quantifiers in Φ_{pfx}. This conversion, called *Skolemization*, is satisfiability preserving. Essentially a QBF Φ is true if and only if its Skolem functions exist such that substituting F_{y_i} for every appearance of y_i in Φ_{mtx} makes the new formula true (i.e., a tautology).

Example 1. Skolemizing the QBF

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2.(x_1 \vee y_1 \vee \neg y_2)(\neg x_1 \vee \neg x_2 \vee y_2)$$

yields

$$\exists F_{y_1} \exists F_{y_2} \forall x_1 \forall x_2.(x_1 \vee F_{y_1} \vee \neg F_{y_2})(\neg x_1 \vee \neg x_2 \vee F_{y_2})$$

where F_{y_1} is a 1-ary function symbol referring to x_1, and F_{y_2} is a 2-ary function symbol referring to x_1 and x_2. Since the QBF is true, Skolem functions exist, for instance, $F_{y_1} = \neg x_1$ and $F_{y_2} = x_1 \wedge x_2$.

The notion of Skolem function has its dual form, known as the *Herbrand function.* For a QBF Φ, the Herbrand function F_{x_i} of variable $x_i \in X$ refers only to the existential variables $y_j \in Y$ with $\ell(y_j) < \ell(x_i)$. Essentially a QBF Φ is false if and only if Herbrand functions exist such that substituting F_{x_i} for every appearance of x_i in Φ_{mtx} makes the new formula false (i.e., unsatisfiable) [3].

2.2 Dependency Quantified Boolean Formulae

A *dependency quantified Boolean formula* (DQBF) generalizes a QBF in its allowance for explicit specification of variable dependencies. Syntactically, a DQBF Φ is the same as a QBF except that in Φ_{pfx} an existential variable y_i is annotated with the set $S_i \subseteq X$ of universal variables referred to by its Skolem function, denoted as $\exists y_{i(S_i)}$, or a universal variable x_j is annotated with the set $H_j \subseteq Y$ of existential variables referred to by its Herbrand function, denoted as $\forall x_{j(H_j)}$, where S_i and H_j are called the *support sets* of y_i and x_j, respectively. However, either the dependencies for the existential variables or the dependencies for the universal variables (but not both) shall be specified. That is, a prenex DQBF is in either of the two forms:

$$\text{S-form: } \forall x_1 \cdots \forall x_n \exists y_{1(S_1)} \cdots \exists y_{m(S_m)}.\phi \qquad (1)$$

$$\text{H-form: } \forall x_{1(H_1)} \cdots \forall x_{n(H_n)} \exists y_1 \cdots \exists y_m.\phi \qquad (2)$$

where ϕ is some quantifier-free formula. Note that *the syntactic quantification order in the prefix of a DQBF is immaterial and can be arbitrary* because the variable dependencies are explicitly specified by the support sets. Such quantification with dependency specification corresponds to the Henkin quantifier [9].[1]

By the above syntactic extension of DQBF, the inputs of the Skolem (respectively Herbrand) function of an existential (respectively universal) variable can be explicitly specified, rather than inferred from the syntactic quantification order. That is, an existential variable y_i (respectively universal variable x_j) can be specified to be semantically independent of a universal variable (respectively an existential variable) whose syntactic scope covers y_i (respectively x_j). Unlike the totally ordered set formed by those of a QBF, the support sets of the existential or universal variables of a DQBF form a partially ordered set in general. This extension makes DQBF more succinct in expressive power than QBF [11].

For the semantics, the truth and falsity of a DQBF can be interpreted by the existence of Skolem and Herbrand functions. Precisely an S-form (respectively H-form) DQBF is true (respectively false) if and only if its Skolem (respectively Herbrand) functions exist for the existential (respectively universal) variables while the specified variable dependencies are satisfied. Consequently, Skolem functions serve as the model to a true S-form DQBF whereas Herbrand functions serve as the countermodel to a false H-form DQBF.

Alternatively, the truth and falsity of a DQBF can be understood from a game-theoretic viewpoint. Essentially an S-form DQBF can be interpreted as a game played by one $\forall$-player and m noncooperative $\exists$-players [11]. An S-form DQBF is true if and only if the $\exists$-players have winning strategies, which correspond to the Skolem functions. Similarly an H-form DQBF can be interpreted as a game played by one $\exists$-player and n noncooperative $\forall$-players. An H-form DQBF is false if and only if the $\forall$-players have winning strategies, which correspond to the Herbrand functions.

As was shown in [4, 5], an S-form DQBF Φ can be converted to a *logically equivalent*[2] QBF Φ' by formula expansion on the universal variables. Assume that universal variable x_1 is to be expanded in Formula (1) and $x_1 \notin S_1 \cup \cdots \cup S_{k-1}$ and $x_1 \in S_k \cap \cdots \cap S_m$. Then Formula (1) can be expanded to

$$\forall x_2 \cdots \forall x_n \exists y_{1(S_1)} \cdots \exists y_{k-1(S_{k-1})}$$
$$\exists y_{k(S_k[x_1/0])} \exists y_{k(S_k[x_1/1])} \cdots \exists y_{m(S_m[x_1/0])} \exists y_{m(S_m[x_1/1])} . \phi|_{x_1=0} \wedge \phi|_{x_1=1},$$

where $S_i[x_1/v]$ denotes x_1 in S_i is substituted with logic value $v \in \{0, 1\}$, and $\phi|_{x_1=v}$ denotes all appearances of x_1 in ϕ are substituted with v including those in the support sets of variables $y_{i(S_i)}$ for $i = k, \ldots, m$. (The subscript of the support set of an existential variable are helpful for tracing expansion paths. Different expansion paths of an existential variable result in distinct existential variables.) Such expansion can be repeatedly applied for every universal variables. The resultant formula after expanding all universal variables is a QBF,

[1] Henkin quantifiers in their original proposal [9] specify dependencies for existential variables only. The dependencies are extended in this paper to universal variables.

[2] That is, Φ and Φ' characterize the same set of Skolem-function models (by properly relating the existential variables of Φ' to those of Φ).

whose variables are all existentially quantified. As to be shown in Section 3.2, expansion can be applied also to H-form DQBF.

3 Properties of DQBF

3.1 Negation vs. Complement

In the light of QBF certification, where there always exists either a Skolem-function model or a Herbrand-function countermodel to a QBF, one intriguing question is whether or not the same property carries to DQBF as well. To answer this question, we distinguish two operators, *negation* (symbolized by "¬") and *complement* (by "∼"), for DQBF. Let Φ_S and Φ_H be Formulae (1) and (2), respectively. By negation, we define

$$\neg\Phi_S = \exists x_1 \cdots \exists x_n \forall y_{1(S_1)} \cdots \forall y_{m(S_m)}.\neg\phi \text{ and} \tag{3}$$

$$\neg\Phi_H = \exists x_{1(H_1)} \cdots \exists x_{n(H_n)} \forall y_1 \cdots \forall y_m.\neg\phi. \tag{4}$$

By complement, we define

$$\sim\!\Phi_S = \exists x_{1(H'_1)} \cdots \exists x_{n(H'_n)} \forall y_1 \cdots \forall y_m.\neg\phi \text{ and} \tag{5}$$

$$\sim\!\Phi_H = \exists x_1 \cdots \exists x_n \forall y_{1(S'_1)} \cdots \forall y_{m(S'_m)}.\neg\phi, \tag{6}$$

where $H'_i = \{y_j \in Y \mid x_i \notin S_j\}$ and $S'_k = \{x_l \in X \mid y_k \notin H_l\}$, which follow what we call the *complementary principle* of the Skolem and Herbrand support sets.

By the above definitions, one verifies that $\neg\neg\Phi = \Phi$, $\sim\sim\Phi = \Phi$, and $\neg\sim\Phi = \sim\neg\Phi$. Moreover, because the Skolem functions of Φ_S, if they exist, are exactly the Herbrand functions of $\neg\Phi_S$, and the Herbrand functions of Φ_H, if they exist, are exactly the Skolem functions of $\neg\Phi_H$, the following proposition holds.

Proposition 1. *DQBF under the negation operation obey the* law of the excluded middle*. That is, a DQBF is true if and only if its negation is false.*

Since any DQBF can be converted to a logically equivalent QBF by formula expansion, it also explains that the law of the excluded middle should hold under negation for DQBF as it holds for QBF.

A remaining question is whether or not the complement of DQBF obeys the law of the excluded middle. The answer to this question is in general negative as we show below. Based on the existence of Skolem and Herbrand functions, we classify DQBF into four categories:

$$\begin{aligned}
\mathcal{C}_S &= \{\Phi \mid \Phi \text{ is true and } \sim\!\Phi \text{ is false}\},\\
\mathcal{C}_H &= \{\Phi \mid \Phi \text{ is false and } \sim\!\Phi \text{ is true}\},\\
\mathcal{C}_{SH} &= \{\Phi \mid \Phi \text{ and } \sim\!\Phi \text{ are true for S-form } \Phi, \text{ or false for H-form } \Phi\}, \text{ and}\\
\mathcal{C}_\emptyset &= \{\Phi \mid \Phi \text{ and } \sim\!\Phi \text{ are false for S-form } \Phi, \text{ or true for H-form } \Phi\}.
\end{aligned}$$

Note that if $\Phi \in \mathcal{C}_S$, then $\sim\!\Phi \in \mathcal{C}_H$; if $\Phi \in \mathcal{C}_H$, then $\sim\!\Phi \in \mathcal{C}_S$; if $\Phi \in \mathcal{C}_{SH}$, then $\sim\!\Phi \in \mathcal{C}_{SH}$; if $\Phi \in \mathcal{C}_\emptyset$, then $\sim\!\Phi \in \mathcal{C}_\emptyset$.

Under the above DQBF partition, observe that the complement of DQBF obeys the law of the excluded middle if and only if $\mathcal{C}_{SH}$ and $\mathcal{C}_\emptyset$ are empty. In fact, as to be shown, for any QBF Φ, $\Phi \notin \mathcal{C}_{SH} \cup \mathcal{C}_\emptyset$. As a consequence, the complement and negation operations for any QBF Φ coincide, and thus $\neg{\sim}\Phi = \Phi$. However, for general DQBF, $\mathcal{C}_{SH}$ and $\mathcal{C}_\emptyset$ are not empty as the following two examples show.

Example 2. Consider the DQBF

$$\Phi = \forall x_1 \forall x_2 \exists y_{1(x_1)} \exists y_{2(x_2)}.((y_1 \oplus x_1) \wedge (y_2 \overline{\oplus} x_2)) \vee ((y_2 \oplus x_2) \wedge (y_1 \overline{\oplus} x_1)),$$

where symbols "$\oplus$" and $\overline{\oplus}$ stand for Boolean XOR and XNOR operators, respectively. Φ has Skolem functions, e.g., x_1 and $\neg x_2$ for existential variables y_1 and y_2, respectively, and $\neg{\sim}\Phi$ has Herbrand functions, e.g., y_2 and y_1 for universal variables x_1 for x_2, respectively. That is, $\Phi \in \mathcal{C}_{SH}$.

Example 3. Consider the DQBF

$$\Phi = \forall x_1 \forall x_2 \exists y_{1(x_1)} \exists y_{2(x_2)}.(y_1 \vee \neg x_1 \vee x_2) \wedge (y_2 \vee x_1 \vee \neg x_2) \wedge (\neg y_1 \vee \neg y_2 \vee \neg x_1 \vee \neg x_2).$$

It can be verified that Φ has no Skolem functions, and $\neg{\sim}\Phi$ has no Herbrand functions. That is, $\Phi \in \mathcal{C}_\emptyset$.

By these two examples, the following proposition can be concluded.

Proposition 2. *DQBF under the complement operation do not obey the law of the excluded middle. That is, the truth (falsity) of a DQBF cannot be decided from the falsity (truth) of its complement.*

Nevertheless, if a DQBF $\Phi \notin \mathcal{C}_{SH} \cup \mathcal{C}_\emptyset$, then its truth and falsity can surely be certified by a Skolem-function model and a Herbrand-function countermodel, respectively.[3] That is, excluding $\Phi \in \mathcal{C}_{SH} \cup \mathcal{C}_\emptyset$, DQBF under the complement operation obeys the law of the excluded middle.

A sufficient condition for a DQBF not in $\mathcal{C}_{SH}$ (equivalently, a necessary condition for a DQBF in $\mathcal{C}_{SH}$) is presented in Theorem 1.

Theorem 1. *Let ϕ be a quantifier-free formula over variables $X \cup Y$, let $\Phi_1 = \forall x_1 \cdots \forall x_n \exists y_{1(S_1)} \cdots \exists y_{m(S_m)}.\phi$ and $\Phi_2 = \forall x_{1(H_1)} \cdots \forall x_{n(H_n)} \exists y_1 \cdots \exists y_m.\phi$ with $H_i = \{y_j \in Y \mid x_i \notin S_{y_j}\}$. Then there exist Skolem functions $\boldsymbol{f} = (f_1, \ldots, f_m)$ for Φ_1 and Herbrand functions $\boldsymbol{g} = (g_1, \ldots, g_n)$ for Φ_2 only if the composite function vector $\boldsymbol{g} \circ \boldsymbol{f}$ admits no fixed-point, that is, there exists no truth assignment α to variables $\boldsymbol{x} = (x_1, \ldots, x_n)$ such that $\alpha = \boldsymbol{g}(\boldsymbol{f}(\alpha))$.*

Proof. Since Φ_1 is true and has Skolem functions $\boldsymbol{f}$, formula $\phi[\boldsymbol{y}/\boldsymbol{f}]$ must be a tautology. On the other hand, since Φ_2 is false and has Herbrand functions $\boldsymbol{g}$, formula $\phi[\boldsymbol{x}/\boldsymbol{g}]$ must be unsatisfiable. Suppose that the fixed-point condition $\alpha = \boldsymbol{g}(\boldsymbol{f}(\alpha))$ holds under some truth assignment α to $\boldsymbol{x}$. Then $\phi[\boldsymbol{y}/\boldsymbol{f}]|_\alpha = \phi[\boldsymbol{x}/\boldsymbol{g}]|_\beta$ for $\beta = \boldsymbol{f}(\alpha)$ being the truth assignment to $\boldsymbol{y}$. It contradicts with the fact that $\phi[\boldsymbol{y}/\boldsymbol{f}]$ must be a tautology and $\phi[\boldsymbol{x}/\boldsymbol{g}]$ must be unsatisfiable. ∎

[3] In general a false S-form DQBF has no Herbrand-function countermodel, and a true H-form DQBF has no Skolem-function model.

The following corollary shows that $\Phi \notin \mathcal{C}_{SH}$ for any QBF Φ.

Corollary 1. *For any QBF Φ, the Skolem-function model and Herbrand-function countermodel cannot co-exist.*

Proof. If a QBF is false, its Skolem-function model does not exist and the corollary trivially holds. Without loss of generality, assume a true QBF is of the form $\Phi = \exists \boldsymbol{y_1} \forall \boldsymbol{x_1} \cdots \exists \boldsymbol{y_n} \forall \boldsymbol{x_n}.\phi$. Let $\{\boldsymbol{y_1} = \boldsymbol{f_1}(), \ldots, \boldsymbol{y_n} = \boldsymbol{f_n}(\boldsymbol{x_1}, \ldots, \boldsymbol{x_{n-1}})\}$ be a model for Φ. Further by contradiction assume there exist a countermodel $\{\boldsymbol{x_1} = \boldsymbol{g_1}(\boldsymbol{y_1}), \ldots, \boldsymbol{x_n} = \boldsymbol{g_n}(\boldsymbol{y_1}, \ldots, \boldsymbol{y_n})\}$. So the fixed-point condition is $\{\boldsymbol{x_1} = \boldsymbol{g_1}(\boldsymbol{f_1}()), \ldots, \boldsymbol{x_n} = \boldsymbol{g_n}(\boldsymbol{f_1}(), \ldots, \boldsymbol{f_n}(\boldsymbol{x_1}, \ldots, \boldsymbol{x_{n-1}}))\}$. Since no cyclic dependency presents in the fixed-point equations, the set of equations always has a solution. In other words, due to the complete ordering of the prefix of a QBF, a fixed-point exists. By Theorem 1, the Skolem-function model and Herbrand-function countermodel cannot co-exist. ■

A sufficient condition for a DQBF not in $\mathcal{C}_\emptyset$ can be characterized by procedure *HerbrandConstruct* as shown in Figure 1. Note that although the algorithm computes Herbrand functions of $\neg{\sim}\Phi_S$ for a false S-form DQBF Φ_S, it can be used to compute Skolem functions of $\neg{\sim}\Phi_H$ for a true H-form DQBF Φ_H by taking as input the negation of the formula.

Given a false S-form DQBF Φ with $n \geq 1$ universal variables, procedure *HerbrandConstruct* in line 1 collects the support set H_n for universal variable x_n. Let $H_n = \{y_{a_1}, \ldots, y_{a_k}\}$ and the rest be $\{y_{a_{k+1}}, \ldots, y_{a_m}\}$. It then recursively constructs the Herbrand functions of the formula expanded on x_n until $n = 1$. By formula expansion on x_n in line 3, variables $\{y_{a_{k+1}}, \ldots, y_{a_m}\}$, which depend on x_n, are instantiated in Φ_{exp} into two copies, say, $\{y'_{a_{k+1}}, y''_{a_{k+1}}, \ldots, y'_{a_m}, y''_{a_m}\}$. Then the *VariableMerge* step in line 6 lets $g_i = g_i^\dagger[y'_{a_{k+1}}/y_{a_{k+1}}, y''_{a_{k+1}}/y_{a_{k+1}}, \ldots, y'_{a_m}/y_{a_m}, y''_{a_m}/y_{a_m}]$.[4] In constructing the Herbrand function g_n of x_n, each assignment α to H_n is examined. Since Herbrand function aims to falsity ϕ, the value of $g_n(\alpha)$ is set to the x_n value that makes $\phi[x_1/g_1, \ldots, x_{n-1}/g_{n-1}]|_\alpha$ unsatisfiable.

Theorem 2. *Given a false S-form DQBF Φ, algorithm HerbrandConstruct returns either nothing or correct Herbrand functions, which falsify $\neg{\sim}\Phi$.*

Proof. Observe first that the functions returned by the algorithm satisfy the support-set dependencies for the universal variables. It remains to show that $\phi[x_1/g_1, \ldots, x_n/g_n]$ is unsatisfiable. By contradiction, suppose there exists an assignment β to the existential variables Y such that $\phi[x_1/g_1, \ldots, x_n/g_n]|_\beta = 1$. Let $v \in \{0, 1\}$ be the value of $g_n|_\alpha$ for α being the projection of β on $H_n \subseteq Y$. Then $\phi[x_1/g_1, \ldots, x_{n-1}/g_{n-1}, x_n/v]|_\beta = 1$. However it contradicts with the way

[4] The method to perform *VariableMerge* in line 6 is not unique. In theory, as long as no violation of variable dependencies is incurred, any substitution can be applied. In practice, however the choice of substitution may affect the strength of the algorithm *HerbrandConstruct* in terms of the likelihood of returning (non-empty) Herbrand functions.

HerbrandConstruct

input: a false S-form DQBF $\Phi = \forall x_1 \cdots \forall x_n \exists y_{1(S_1)} \cdots \exists y_{m(S_m)}.\phi$, and the number n of universal variables

output: Herbrand-functions $(g_1, \cdots, g_n)$ of $\neg\sim\Phi$

01 $H_n := \{y_i \in Y \mid x_n \notin S_i\}$

02 **if** $(n > 1)$

03 $\quad \Phi_{\text{exp}} :=$ *FormulaExpand*(Φ, x_n);

04 $\quad \boldsymbol{g}^\dagger :=$ *HerbrandConstruct*$(\Phi_{\text{exp}}, n-1)$;

05 $\quad$ **if** $(\boldsymbol{g}^\dagger = \emptyset)$ **return** $\emptyset$;

06 $\quad \boldsymbol{g} :=$ *VariableMerge*$(\boldsymbol{g}^\dagger)$;

07 $\quad$ **for each** assignment α to H_n

08 $\quad\quad$ **if** $(\phi[x_1/g_1, \ldots, x_{n-1}/g_{n-1}]|_{\alpha, x_n=0}$ is unsatisfiable)

09 $\quad\quad\quad g_n(\alpha) = 0$;

10 $\quad\quad$ **if** $(\phi[x_1/g_1, \ldots, x_{n-1}/g_{n-1}]|_{\alpha, x_n=1}$ is unsatisfiable)

11 $\quad\quad\quad g_n(\alpha) = 1$;

12 $\quad\quad$ **else return** $\emptyset$;

13 **else**

14 $\quad$ **for each** assignment α to H_n

15 $\quad\quad$ **if** $(\phi|_{\alpha, x_n=0}$ is unsatisfiable)

16 $\quad\quad\quad g_n(\alpha) = 0$;

17 $\quad\quad$ **if** $(\phi|_{\alpha, x_n=1}$ is unsatisfiable)

18 $\quad\quad\quad g_n(\alpha) = 1$;

19 $\quad\quad$ **else return** $\emptyset$;

20 **return** $(g_1, \ldots, g_n)$;

end

Fig. 1. Algorithm: Herbrand-function Construction

how $g_n|_\alpha$ is constructed. Hence the returned Herbrand functions $(g_1, \ldots, g_n)$, if they are not empty, are indeed correct Herbrand functions. ∎

The following corollary shows that $\Phi \notin \mathcal{C}_\emptyset$ for any QBF Φ.

Corollary 2. *If Φ is a false QBF and its universal variables $x_1, \ldots, x_n$ follow the QBF's prefix order, algorithm HerbrandConstruct always returns non-empty Herbrand functions.*

Proof. We prove the statement by induction on the number of universal variables. For the base case, without loss of generality consider QBF $\Phi = \exists y_1 \cdots \exists y_k \forall x \exists y_{k+1} \cdots \exists y_m.\phi$. After line 1, *HerbrandConstruct* enters line 14. Since the QBF is false and has only one universal variable x, expanding on x yields a purely existentially quantified unsatisfiable formula: $\exists y_1 \cdots \exists y_k (\exists y'_{k+1} \cdots \exists y'_m.\phi|_{x=0} \wedge \exists y''_{k+1} \cdots \exists y''_m.\phi|_{x=1})$. By its unsatisfiability, for every assignment α to $y_1, \cdots, y_k$, formula $\exists y'_{k+1} \cdots \exists y'_m.\phi|_{\alpha,x=0} \wedge \exists y''_{k+1} \cdots \exists y''_m.\phi|_{\alpha,x=1}$ must be unsatisfiable. Since $\exists y'_{k+1} \cdots \exists y'_m.\phi|_{\alpha,x=0}$ and $\exists y''_{k+1} \cdots \exists y''_m.\phi|_{\alpha,x=1}$ share no common variables, at least one of them must be unsatisfiable. Hence the procedure returns a non-empty Herbrand function.

For the inductive step, assume the previous recursive calls for $k = 1, \ldots, n-1$ of *HerbrandConstruct* do not return $\emptyset$. We show that the current call for $k = n$ cannot return $\emptyset$. Expanding Φ on x_n yields $\Phi_{\text{exp}} = \forall x_1 \cdots \forall x_{n-1} \exists y_{1(S_1)} \cdots \exists y_{k(S_k)}$ $(\exists y'_{k+1(S_{k+1})} \cdots \exists y'_{m(S_m)}.\phi|_{x_n=0} \wedge \exists y''_{k+1(S_{k+1})} \cdots \exists y''_{m(S_m)}.\phi|_{x_n=1})$. By the inductive hypothesis, functions $g_1^{\dagger}, \cdots, g_{n-1}^{\dagger}$ are returned. Moreover, $g_i^{\dagger}$ for any $i = 1, \ldots, n-1$ is independent of y'_j and y''_j for $j = k+1, \ldots, m$. So we construct $g_i = g_i^{\dagger}$. Since $g_1, \ldots, g_{n-1}$ have been constructed in a way such that $\exists y_1 \cdots \exists y_k (\exists y'_{k+1} \cdots \exists y'_m.\phi[x_1/g_1, \cdots, x_{n-1}/g_{n-1}]|_{x_n=0} \wedge \exists y''_{k+1} \cdots \exists y''_m.\phi[x_1/g_1, \cdots, x_{n-1}/g_{n-1}]|_{x_n=1})$ is unsatisfiable, under every assignment α to $y_1, \cdots, y_k$ formula $(\exists y'_{k+1} \cdots \exists y'_m.\phi[x_1/g_1, \cdots, x_{n-1}/g_{n-1}]|_{x_n=0} \wedge \exists y''_{k+1} \cdots \exists y''_m.\phi[x_1/g_1, \cdots, x_{n-1}/g_{n-1}]|_{x_n=1})$ is unsatisfiable. Moreover, since $\exists y'_{k+1} \cdots \exists y'_m.\ \phi[x_1/g_1, \cdots, x_{n-1}/g_{n-1}]|_{x_n=0}$ and $\exists\, y''_{k+1} \cdots \exists y''_m.\ \phi[x_1/g_1, \cdots, x_{n-1}/g_{n-1}]|_{x_n=1}$ do not share any variables, at least one of them must be unsatisfiable. So g_n is returned. ■

Note that the above proof does not explicitly perform the substitution $g_i = g_i^{\dagger}[y'_{a_{k+1}}/y_{a_{k+1}}, y''_{a_{k+1}}/y_{a_{k+1}}, \ldots, y'_{a_m}/y_{a_m}, y''_{a_m}/y_{a_m}]$ in *VariableMerge* because all g_i in fact do not depend on primed or double-primed variables in the QBF case.

Procedure *HerbrandConstruct* is useful in deriving Herbrand functions not only for QBF but also for general DQBF as the following example suggests.

Example 4. Consider the DQBF $\Phi = \forall x_1 \forall x_2 \exists y_{1(x_1)} \exists y_{2(x_2)}.\phi$ with $\phi = (y_1 \vee x_2) \wedge (y_2 \vee x_1) \wedge (\neg y_1 \vee \neg y_2 \vee \neg x_1 \vee \neg x_2)$. *HerbrandConstruct*$(\Phi, 2)$ computes Herbrand functions for $\neg{\sim}\Phi$ with the following steps. Expanding Φ on x_2 yields $\Phi_{\text{exp}} = \forall x_1 \exists y_{1(x_1)} \exists y'_2 \exists y''_2.\phi|_{x_2=0} \wedge \phi|_{x_2=1}$ with $\phi|_{x_2=0} = (y_1) \wedge (y'_2 \vee x_1)$ and $\phi|_{x_2=1} = (y''_2 \vee x_1) \wedge (\neg y_1 \vee \neg y''_2 \vee \neg x_1)$. The recursive call to *HerbrandConstruct*$(\Phi_{\text{exp}}, 1)$ determines the value of function $g_1^{\dagger}(y'_2, y''_2)$ under every assignment α to (y'_2, y''_2). In particular, $g_1^{\dagger}(0,0) = 0$ due to $\phi_{\text{exp}} = (y_1) \wedge (x_1) \wedge (x_1)$; $g_1^{\dagger}(0,1) = 0$ (or 1) due to $\phi_{\text{exp}} = (y_1) \wedge (x_1) \wedge (\neg y_1 \vee \neg x_1)$; $g_1^{\dagger}(1,0) = 0$ due to $\phi_{\text{exp}} = (y_1) \wedge (x_1)$; $g_1^{\dagger}(1,1) = 1$ due to $\phi_{\text{exp}} = (y_1) \wedge (\neg y_1 \vee \neg x_1)$. So $g_1^{\dagger}(y'_2, y''_2) = y'_2 y''_2$ (or y''_2), and $g_1(y_2) = g_1^{\dagger}[y'_2/y_2, y''_2/y_2] = y_2$.

Returning to *HerbrandConstruct*$(\Phi, 2)$, we have $\phi[x_1/g_1] = (y_1 \vee x_2) \wedge (y_2) \wedge (\neg y_1 \vee \neg y_2 \vee \neg x_2)$. The value of function g_2 for each assignment α to y_1 can be determined with $g_2(0) = 0$ due to $\phi[x_1/g_1]|_{y_1=0} = (x_2) \wedge (y_2)$ and $g_2(1) = 1$ due to $\phi[x_1/g_1]|_{y_1=1} = (y_2) \wedge (\neg y_2 \vee \neg x_2)$. That is, $g_2(y_1) = y_1$. The computed g_1 and g_2 indeed make $\phi[x_1/g_1, x_2/g_2] = (y_1) \wedge (y_2) \wedge (\neg y_1 \vee \neg y_2)$ unsatisfiable.

Since the DQBF subset $\mathcal{C}_S \cup \mathcal{C}_H$ obeys the law of the excluded middle under the complement operation, Theorems 1 and 2 provide a tool to test whether a DQBF Φ can be equivalently expressed as $\neg{\sim}\Phi$, that is, whether a DQBF has either a Skolem-function model or a Herbrand-function countermodel. Figure 2 shows the four DQBF categories and the regions characterized by Theorems 1 and 2.

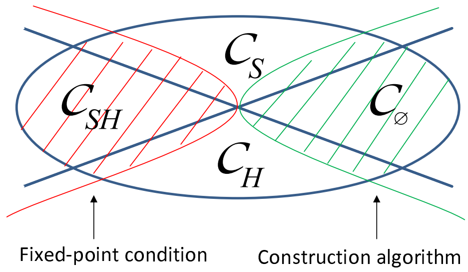

Fig. 2. Four DQBF categories and regions characterized by Theorems 1 and 2.

3.2 Formula Expansion on Existential Variables

Formula expansion on existential variables for DQBF can be achieved by negation using De Morgan's law and expansion on universal variables. It leads to the following expansion rule, which is dual to expanding universal variables.

Proposition 3. *Given a DQBF* $\forall x_{1(H_1)} \cdots \forall x_{n(H_n)} \exists y_1 \cdots \exists y_m.\phi$, *assume without loss of generality that* y_1 *is to be expanded with* $y_1 \notin H_1 \cup \cdots \cup H_{k-1}$ *and* $y_1 \in H_k \cap \cdots \cap H_n$. *The formula can be expanded to*

$$\forall x_{1(H_1)} \cdots \forall x_{k-1(H_{k-1})} \forall x_{k(H_k[y_1/0])} \forall x_{k(H_k[y_1/1])} \cdots \forall x_{n(H_n[y_1/0])} \forall x_{n(H_n[y_1/1])}$$
$$\exists y_2 \cdots \exists y_m.\phi|_{y_1=0} \vee \phi|_{y_1=1},$$

where $H_i[y_1/v]$ *denotes* y_1 *in* H_i *is substituted with logic value* $v \in \{0,1\}$, *and* $\phi|_{y_1=v}$ *denotes all appearances of* y_1 *in* ϕ *are substituted with* v *including those in the support sets of variables* $x_{i(H_i)}$ *for* $i = k, \ldots, n$.

Such expansion can be repeatedly applied for every existential variables. The resultant formula after expanding all existential variables is a QBF. Note that, when Skolem functions are concerned rather than Herbrand functions, the support sets of the existential variables should be listed and can be obtained from H_i by the aforementioned complementary principle.

Example 5. Consider expanding variable y_1 of DQBF

$$\Phi = \forall x_{1(y_1)} \forall x_{2(y_2)} \forall x_{3(y_3)} \exists y_1 \exists y_2 \exists y_3.\phi.$$

By De Morgan's law and expansion on a universal variable, we obtain

$$\begin{aligned} \neg\neg\Phi &= \neg\exists x_{1(y_1)} \exists x_{2(y_2)} \exists x_{3(y_3)} \forall y_1 \forall y_2 \forall y_3.\neg\phi \\ &= \neg\exists x_{1(0)} \exists x_{1(1)} \exists x_{2(y_2)} \exists x_{3(y_3)} \forall y_2 \forall y_3.\neg\phi|_{y_1=0} \wedge \neg\phi|_{y_1=1} \\ &= \forall x_{1(0)} \forall x_{1(1)} \forall x_{2(y_2)} \forall x_{3(y_3)} \exists y_2 \exists y_3.\phi|_{y_1=0} \vee \phi|_{y_1=1}. \end{aligned}$$

3.3 Prenex and Non-prenex Conversion

This section studies some syntactic rules that allow localization of quantifiers to sub-formulae. We focus on the truth (namely the Skolem-function model), while similar results can be concluded by duality for the falsity (namely the Herbrand-function countermodel), of a formula.

The following proposition shows the localization of existential quantifiers to the sub-formulas of a disjunction.

Proposition 4. *The DQBF*

$$\forall \boldsymbol{x} \exists y_{1(S_1)} \cdots \exists y_{m(S_m)}. \phi_A \vee \phi_B,$$

where $\forall \boldsymbol{x}$ *denotes* $\forall x_1 \cdots \forall x_n$*, sub-formula* ϕ_A *(respectively* ϕ_B*) refers to variables* $X_A \subseteq X$ *and* $Y_A \subseteq Y$ *(respectively* $X_B \subseteq X$ *and* $Y_B \subseteq Y$*), is logically equivalent to*

$$\forall \boldsymbol{x_c} \left(\forall \boldsymbol{x_a} \exists y_{a_1(S_{a_1} \cap X_A)} \cdots \exists y_{a_p(S_{a_p} \cap X_A)} \phi_A \vee \forall \boldsymbol{x_b} \exists y_{b_1(S_{b_1} \cap X_B)} \cdots \exists y_{b_q(S_{b_q} \cap X_B)} \phi_B \right),$$

where variables $\boldsymbol{x_c}$ *are in* $X_A \cap X_B$*, variables* $\boldsymbol{x_a}$ *are in* $X_A \backslash X_B$*, variables* $\boldsymbol{x_b}$ *are in* $X_B \backslash X_A$*,* $y_{a_i} \in Y_A$*, and* $y_{b_j} \in Y_B$*.*

Proof. A model to the former expression consists of every truth assignment to X and the induced Skolem function valuation to Y. Since every such combined assignment to $X \cup Y$ either satisfies ϕ_A or ϕ_B, by collecting those satisfying ϕ_A (respectively ϕ_B) and projecting to variables $X_A \cup Y_A$ (respectively $X_B \cup Y_B$) the model (i.e., the Skolem functions for $\boldsymbol{y_a}$ and $\boldsymbol{y_b}$) to the latter expression can be constructed. (Note that, for a quantifier $\exists y_i$ splitting into two, one for ϕ_A and the other for ϕ_B, in the latter expression, they are considered distinct and have their own Skolem functions.)

In addition, the Skolem functions for $\forall \boldsymbol{x_a} \exists y_{a_1(S_{a_1} \cap X_A)} \cdots \exists y_{a_p(S_{a_p} \cap X_A)} \phi_A|_\alpha$ and those for $\forall \boldsymbol{x_b} \exists y_{b_1(S_{b_1} \cap X_B)} \cdots \exists y_{b_q(S_{b_q} \cap X_B)} \phi_B|_\alpha$ under every assignment α to $\boldsymbol{x_c}$ can be collected and combined to form a model for the former expression. In particular the respective Skolem functions $f_{a_j}|_\alpha$ and $f_{b_k}|_\alpha$ under α for y_{a_j} and y_{b_k} originating from the same quantifier y_i in the former expression are merged into one Skolem function $f_i = \bigvee_\alpha \left(\chi_\alpha (f_{a_j}|_\alpha \vee f_{b_k}|_\alpha) \right)$, where χ_α denotes the characteristic function of α, e.g., $\chi_\alpha = x_1 x_2 \neg x_3$ for $\alpha = (x_1 = 1, x_2 = 1, x_3 = 0)$. ■

Example 6. Consider the QBF

$$\Phi = \forall x_1 \exists y_1 \forall x_2 \exists y_2 \forall x_3 \exists y_3 . \phi_A \vee \phi_B$$

with ϕ_A refers to variables x_1, x_2, y_1, y_2 and ϕ_B refers to x_2, x_3, y_2, y_3. It has the following equivalent DQBF expressions.

$$\begin{aligned}
\Phi &= \forall x_1 \forall x_2 \forall x_3 \exists y_{1(x_1)} \exists y_{2(x_1,x_2)} \exists y_{3(x_1,x_2,x_3)} . \phi_A \vee \phi_B \\
&= \forall x_1 \forall x_2 \forall x_3 \left(\exists y_{1(x_1)} \exists y_{2(x_1,x_2)} \phi_A \vee \exists y_{2(x_2)} \exists y_{3(x_2,x_3)} \phi_B \right) \\
&= \forall x_2 \left(\forall x_1 \exists y_{1(x_1)} \exists y_{2(x_1,x_2)} \phi_A \vee \forall x_3 \exists y_{2(x_2)} \exists y_{3(x_2,x_3)} \phi_B \right)
\end{aligned}$$

In contrast, conventionally the quantifiers of the QBF can only be localized to

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \left(\phi_A \vee \forall x_3 \exists y_3 \phi_B\right).$$

The following proposition shows the localization of universal quantifiers to a sub-formula of a conjunction.

Proposition 5. *The DQBF*

$$\forall \boldsymbol{x} \exists y_{1(S_1)} \cdots \exists y_{k(S_k)} . \phi_A \wedge \phi_B,$$

where $\forall \boldsymbol{x}$ *denotes* $\forall x_1 \cdots \forall x_n$*, sub-formula* ϕ_A *(respectively* ϕ_B*) refers to variables* $X_A \subseteq X$ *and* $Y_A \subseteq Y$ *(respectively* $X_B \subseteq X$ *and* $Y_B \subseteq Y$*), is logically equivalent to*

$$\forall \boldsymbol{x} \exists y_{2(S_2)} \cdots \exists y_{k(S_k)} . \left(\exists y_{1(S_1 \cap X_A)} \phi_A\right) \wedge \phi_B,$$

for $y_1 \notin Y_B$*.*

Proof. The proposition follows from the fact that the Skolem function of y_1 is purely constrained by ϕ_A only, and is the same for both expressions. Note that the former formula is equivalent to $\forall \boldsymbol{x} \exists y_{1(S_1 \cap X_A)} \cdots \exists y_{k(S_k)} . \phi_A \wedge \phi_B$. ■

Essentially DQBF allow tighter localization of quantifier scopes than QBF. On the other hand, converting a non-prenex QBF to the prenex form may incur the size increase of support sets of existential variables due to the linear (or complete order) structure of the prefix. With DQBF, such spurious increase can be eliminated.

4 Applications

Although to date there is no DQBF solver, we note that the framework provided by QBF solver sKizzo [1], which is based on Skolemization, can be easily extended to DQBF solving. A natural application of DQBF is Boolean relation determinization [8, 3] in logic circuit synthesis. Consider a Boolean relation $R(\boldsymbol{x}, \boldsymbol{y})$ as a characteristic function (quantifier-free Boolean formula) specifying the input and output behavior of some (possibly non-deterministic) combinational system with inputs X and outputs Y. To realize the outputs of the system, the Skolem functions of the QBF

$$\forall \boldsymbol{x} \exists \boldsymbol{y} . R(\boldsymbol{x}, \boldsymbol{y})$$

is to be solved. Often the inputs of some output y_i need to be restricted to depend only on a subset of X. This restriction can be naturally described by DQBF. Therefore DQBF can be exploited for topologically constrained logic synthesis [14].

5 Discussions

IF logic [10] with the game-theoretical semantics is known to violate the law of the excluded middle. A simple example is the IF logic formula $\forall x \exists y_{/x}.(x = y)$ for $x, y \in \{0, 1\}$, where $y_{/x}$ indicates the independence of y on x [7]. It assumes that not only y is independent of x, but also is x independent of y. That is, it is equivalent to $\forall x_{()} \exists y_{()}.(x = y)$ in our dependency notation. In a game-theoretic viewpoint, neither the ∃-player nor the ∀-player has a winning strategy. Therefore this formula is neither true nor false, and has no equivalent DQBF since any DQBF can always be expanded into a QBF, whose truth and falsity can be fully determined.

On the other hand, the game-theoretical semantics of IF logic, when extended to DQBF, does not provide a fully meaningful approach to synthesizing Skolem and Herbrand functions. Unlike the unimportance of the syntactic quantification order in our formulation, the semantic game of IF logic should be played with respect to the prefix order. Since different orders correspond to different games, the semantics is not directly useful in our considered synthesis application.

Henkin quantifiers in their original form [9] specified only the dependencies of existential variables on universal variables. Such restricted dependencies were assumed in early IF logic [10] research. As was argued in [7], the dependency of universal variables on existential variables are necessary to accomplish a symmetric treatment on the falsity, in addition to truth, of an IF logic formula. With such extension, IF logic formulae can be closed under negation. However, how the dependencies of existential variables and universal variables relate to each other was not studied. The essential notion of Herbrand functions was missing. In contrast, our formulation on DQBF treats Skolem and Herbrand functions on an equal footing. Unlike [7], we restrict a formula to be of either S-form or H-form, rather than simultaneous specification of dependencies for existential and universal variables. This restriction makes the synthesis of Skolem and Herbrand functions for DQBF more natural.

Prior work [11, 5] assumed DQBF are of S-form only. In [11], a DQBF was formulated as a game played by a ∀-player and multiple noncooperative ∃-players. This game formulation is fundamentally different from that of IF-logic. The winning strategies, if they exist, of the ∃-players correspond to the Skolem functions of the DQBF. This game interpretation can be naturally extended to H-form DQBF.

6 Conclusions

The syntax and semantics of DQBF presented in this paper made DQBF a natural extension of QBF from a certification viewpoint. Basic DQBF properties, including formula negation, complement, expansion, and prenex and non-prenex form conversion, were shown. Our formulation is adequate for applications where Skolem/Herbrand functions are of concern.

Acknowledgments

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References

1. M. Benedetti. Evaluating QBFs via symbolic Skolemization. In *Proc. Int'l Conf. on Logic for Programming, Artificial Intelligence and Reasoning* (LPAR), 2004.
2. A. Blass and Y. Gurevich. Henkin quantifiers and complete problems. *Annals of Pure and Applied Logic*, 32:1-16, 1986.
3. V. Balabanov and J.-H. R. Jiang. Unified QBF certification and its applications. *Formal Methods in System Design*, 2012.
4. U. Bubeck and H. Kleine Büning. Dependency quantified Horn formulas: Models and complexity. In *Proc. Int'l Conf. on Theory and Applications of Satisfiability Testing* (SAT), pp. 198-211, 2006.
5. U. Bubeck. *Model-based Transformations for Quantified Boolean Formulas*. IOS Press, 2010.
6. X. Caicedo, F. Dechesne and T. Janssen. Equivalence and quanftifier rules for logic with imperfect information. *Logic Journal of the IGPL*, 17(1): 91-129, Oxford University Press, 2009.
7. F. Dechesne. *Game, Set, Maths: Formal Investigations into Logic with Imperfect Information*. PhD thesis, Tilburg University, 2005.
8. J.-H. R. Jiang, H.-P. Lin, and W.-L. Hung. Interpolating functions from large Boolean relations. In *Proc. Int'l Conf. on Computer-Aided Design* (ICCAD), pp. 770-784, 2009.
9. L. Henkin. Some remarks on infinitely long formulas. *Infinitistic Methods*, pp. 167-183, 1961.
10. J. Hintikka and G. Sandu. Informational independence as a semantical phenomenon. In *Logic, Methodology and Philosophy of Science*, pp. 571-589, 1989.
11. G. Peterson, J. Reif, and S. Azhar. Lower bounds for multiplayer non-cooperative games of imcomplete information. *Computers and Mathematics with Applications*, 41(7-8):957-992, 2001.
12. S. Peters and D. Westerstahl. *Quantifiers in Language and Logic*. Oxford University Press, 2006.
13. Th. Skolem. *Uber die mathematische Logik. Norsk. Mat. Tidsk.*, 10:125-142, 1928. [Translation in *From Frege to Gödel, A Source Book in Mathematical Logic*, J. van Heijenoort, Harvard Univ. Press, 1967.]
14. S. Sinha, A. Mishchenko, and R. K. Brayton. Topologically constrained logic synthesis. In *Proc. Int'l Conf. on Computer-Aided Design* (ICCAD), pp. 679-686, 2002.