Henkin Quantifiers and Boolean Formulae: A Certification Perspective of DQBF ☆

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Abstract

Henkin quantifiers, when applied on Boolean formulae, yielding the so-called dependency quantified Boolean formulae (DQBFs), offer succinct descriptive power specifying variable dependencies. Despite their natural applications to games with incomplete information, logic synthesis with constrained input dependencies, etc., DQBFs remain a relatively unexplored subject however. This paper investigates their basic properties, including formula negation and complement, formula expansion, prenex and non-prenex form conversions, and resolution. In particular, the proposed DQBF formulation is established from a synthesis perspective concerned with Skolem-function models and Herbrand-function countermodels. Also a generalized resolution rule is shown to be sound, but incomplete, for DQBF evaluation.

Keywords: Henkin quantifier, quantified Boolean formula, Herbrand function, Skolem function, resolution, consensus

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1. Introduction

Henkin quantifiers [16], also known as branching quantifiers among other names, generalize the standard quantification by admitting explicit specifica-

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tion, for an existentially quantified variable, about its dependence on universally quantified variables. In addition to mathematical logic, Henkin quantifiers appear not uncommonly in various contexts, such as natural languages [19], computation [3], game theory [18], and even system design as to be shown in Section 5. They permit the expression of (in)dependence in language, logic and computation, the modelling of incomplete information in noncooperative games, and the specification of partial dependencies among components in system design, which is the main motivation of this work.

When Henkin quantifiers are imposed on first-order logic (FOL) formulae, it results in the formulation of independence-friendly (IF) logic [17], which was shown to be more expressive than first-order logic and exhibit expressive power same as existential second-order logic. However one notable limitation among others of IF logic under the game-theoretical semantics is the violation of the law of the excluded middle, which states either a proposition or its negation is true. Therefore negating a formula can be problematic in terms of truth and falsity. From a game-theoretical viewpoint, it corresponds to undetermined games, where there are cases under which no player has a winning strategy. Moreover, in synthesis applications, the winning strategies of the semantic games do not exactly correspond to Skolem and Herbrand functions, which form the model and countermodel, respectively, of an underlying formula, although syntactic rules for negating IF logic formulae were suggested in [9, 11].

When Henkin quantifiers are imposed on Boolean formulae, it results in the so-called dependency quantified Boolean formulae (DQBFs), whose validity lies in the complexity class of NEXPTIME-complete as was shown in [18] through the formulation of multiplayer noncooperative games. In contrast to quantified Boolean formulae (QBFs) [22], whose evaluation is PSPACE-complete, DQBFs offer more succinct descriptive power than QBFs provided that NEXPTIME is not in PSPACE. By expansion on universally quantified variables, a DQBF can be converted to a QBF with the cost of exponential blow up in formula size [7, 8].

This paper studies DQBFs from a synthesis perspective. There are applications (such as topologically constrained logic synthesis to be discussed in Section 5), where the Skolem or Herbrand functions of an encoding DQBF directly correspond to desired synthesis targets. For these applications, DQBF certification plays an essential role. Motivated by the conversion of syntactic (Q-consensus/Q-resolution) proofs to semantic (Skolem/Herbrand) certificates in QBFs [5, 6], we investigate DQBF certification. By distinguishing formula negation and complement to account for the duality of variable dependencies arising from the winning strategies of different players, the connection between Skolem and Herbrand functions is established. While the law of the excluded middle holds for negation, it does not hold for complement. The special subset of DQBFs whose truth and falsity coincide with the existence of Skolem and Herbrand functions, respectively, is characterized. Our formulation provides a unified view on DQBF models and countermodels, which encompasses QBFs as a special case. Some fundamental properties of DQBFs are studied in Section 3. In Section 4, the Q-resolution (Q-consensus) rule of QBFs [15] is extended to DQBFs and the resultant resolution (consensus), called DQ-resolution (DQ-consensus), is shown to be sound but incomplete. Application of DQBFs on Boolean relation determinization for input constrained function extraction is discussed in Section 5. Section 6 compares our results with prior work, and finally Section 7 concludes this work.

2. Preliminaries

As conventional notation, a set is denoted with an upper-case letter, e.g., V; its elements are in lower-case letters, e.g., $v_i \in V$. The ordered version (i.e., vector) of $V = \{v_1, \ldots, v_n\}$ is denoted as $\vec{v} = (v_1, \ldots, v_n)$.

A *literal* l in a Boolean formula is either a variable (in this case l is in a positive phase) or the negation of a variable (l in a negative phase). In the sequel, the corresponding variable of a literal l is denoted as $var(l)$. A *clause* (respectively *cube*) is a Boolean formula consisting of a disjunction (respectively *conjunction*) of a set of literals. In the sequel, we may alternatively specify a clause/cube as a set of literals. A formula in the *conjunctive normal form* (CNF) is a conjunction of a set of clauses; a formula in the *disjunctive normal form* (DNF) is a disjunction of a set of cubes.

Substituting a term t (respectively a vector of terms $\vec{t} = (t_1, \ldots, t_n)$) for some variable v (respectively a vector of variables $\vec{v} = (v_1, \ldots, v_n)$) in a formula ϕ is denoted as $\phi[v/t]$ (respectively $\phi[\vec{v}/\vec{t}]$ or $\phi[v_1/t_1, \ldots, v_n/t_n]$). A formula ϕ under some truth assignment α to its variables is denoted as $\phi|_\alpha$.

2.1. Quantified Boolean Formulae

A *quantified Boolean formula* (QBF) Φ over variables $V = \{v_1, \ldots, v_k\}$ in the *prenex form* is expressed as

$$Q_1 v_1 \cdots Q_k v_k.\phi,$$

where $Q_1 v_1 \cdots Q_k v_k$, with $Q_i \in \{\exists, \forall\}$, is called the *prefix*, denoted Φ_p ,and ϕ, a quantifier-free formula over variables V, is called the *matrix*, denoted Φ_m, of Φ. A QBF is in the *prenex conjunctive normal form* (PCNF) and *prenex disjunctive normal form* (PDNF) if it is in the prenex form and in addition its matrix is in CNF and DNF, respectively. We call variable v_i in a QBF an *existential variable* if $Q_i = \exists$, or a *universal variable* if $Q_i = \forall$. Similarly, we call literal l an *existential literal* (respectively a *universal literal*) if $var(l)$ is an existential variable (respectively a universal variable). A QBF is of *non-prenex form* if its quantifiers are scattered around the formula without a clean separation between the prefix and the matrix. Unless otherwise said, we shall assume that a QBF is in the prenex form and is totally quantified, i.e., with no free variables. As a notational convention, unless otherwise specified we shall let $X = \{x_1, \ldots, x_n\}$ be the set of universal variables and $Y = \{y_1, \ldots, y_m\}$ existential variables.

Given a QBF Φ over variables V, the *quantification level* $\ell : V \to \mathbb{N}$ of variable $v_i \in V$ is defined to be the number of quantifier alternations between $\exists$ and $\forall$ from the outermost variable to variable v_i in Φ_p, e.g., $\ell(v_1) = \ell(v_2) = 0$, $\ell(v_3) = 1$, and $\ell(v_4) = 2$ for QBF $\exists v_1 \exists v_2 \forall v_3 \exists v_4.\phi$.

Any QBF Φ over variables $X \cup Y$ can be converted into the well-known *Skolem normal form* [20]. In the conversion, every appearance of $y_i \in Y$ in Φ_m

is replaced by its respective newly introduced function symbol F_{y_i} corresponding to the *Skolem function* of y_i, which refers only to the universal variables $x_j \in X$ with $\ell(x_j) < \ell(y_i)$. These function symbols are then existentially quantified before (on the left of) other universal quantifiers in Φ_{p}. This conversion, called *Skolemization*, is satisfiability preserving. Essentially a QBF Φ is true if and only if its Skolem functions exist such that substituting F_{y_i} for every appearance of y_i in Φ_{m} makes the new formula true (i.e., a tautology).

Example 1. *Skolemizing the QBF*

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2.(x_1 \vee y_1 \vee \neg y_2) \wedge (\neg x_1 \vee \neg x_2 \vee y_2)$$

yields

$$\exists F_{y_1} \exists F_{y_2} \forall x_1 \forall x_2.(x_1 \vee F_{y_1} \vee \neg F_{y_2}) \wedge (\neg x_1 \vee \neg x_2 \vee F_{y_2})$$

where F_{y_1} is a 1-ary function symbol referring to x_1, and F_{y_2} is a 2-ary function symbol referring to x_1 and x_2. Since the QBF is true, Skolem functions exist, for instance, $F_{y_1} = \neg x_1$ and $F_{y_2} = x_1 \wedge x_2$.

The notion of Skolem function has its dual form, known as the *Herbrand function*. For a QBF Φ, the Herbrand function F_{x_i} of variable $x_i \in X$ refers only to the existential variables $y_j \in Y$ with $\ell(y_j) < \ell(x_i)$. Essentially a QBF Φ is false if and only if Herbrand functions exist such that substituting F_{x_i} for every appearance of x_i in Φ_{m} makes the new formula false (i.e., unsatisfiable) [5, 6].

In addition to the above semantic forms of QBF certificates, there are also syntactic forms of certificates based on resolution. For a (quantifier-free) CNF formula, a *resolution step* can be defined on two clauses $C_1 = C_1' \vee l$ and $C_2 = C_2' \vee \neg l$, where C_i' is a sub-clause of C_i, as the process of yielding the clause $C_1' \vee C_2'$, where the variable $var(l)$ of literal l is called the *pivot variable* and the resultant clause is called the *resolvent* of the resolution. (We assume that a resolvent clause, as well as any original clause, is non-tautological, namely, it does not contain both positive and negative phase literals of the same variable.) Resolution is a sound and complete approach to test whether or not a CNF formula is satisfiable. Essentially a CNF formula is unsatisfiable if and only if the empty clause can be generated from repeated applications of resolution. For a QBF in PCNF, *Q-resolution* [15] can be similarly defined with the following two modifications. First, only existential variables can be the pivot variables for Q-resolution. Second, a clause is simplified by $\forall$-*reduction*, defined as the process of removing from C a literal $l \in C$ whenever $\ell(l) = \max_{l_i \in C}\{\ell(l_i)\}$ and $var(l) \in X$. Essentially Q-resolution is a sound and complete approach to QBF evaluation.

Theorem 1 ([15]). *A QBF is false (unsatisfiable) if and only if there exists a clause resolution sequence leading to an empty clause.*

By duality, for a DNF formula, a *consensus step* can be defined on two cubes $C_1 = C_1' \wedge l$ and $C_2 = C_2' \wedge \neg l$, where C_i' is a sub-cube of C_i, as the process of yielding the (non-false) cube $C_1' \wedge C_2'$, where the variable $var(l)$ of literal l is called the *pivot variable* and the resultant cube is called the *consensus cube* of the consensus. (We assume that a consensus cube, as well as any original cube, is non-false, namely, it does not contain both positive and negative phase literals of the same variable.) Consensus is a sound and complete approach to test whether or not a DNF formula is a tautology. That is, a DNF formula is a tautology if and only if the empty cube can be generated from repeated applications of consensus. For a QBF in PDNF, *Q-consensus* can be defined in a way similar to Q-resolution. First, only universal variables can be pivot variables for Q-consensus. Second, a cube is simplified by $\exists$-*reduction*, defined as the process of removing from C a literal $l \in C$ whenever $\ell(l) = \max_{l_i \in C}\{\ell(l_i)\}$ and $var(l) \in Y$. Essentially Q-consensus is a sound and complete approach to QBF evaluation. The connection between Q-resolution (respectively Q-consensus) proofs and Herbrand (respectively Skolem) functions was established in [5, 6].

Theorem 2 ([5, 6]). *Given a Q-resolution (respectively Q-consensus) proof of a false (respectively true) QBF, there exists an algorithm that converts the proof to a Herbrand-function countermodel (respectively Skolem-function model) in time linear with respect to the proof size.*

2.2. Dependency Quantified Boolean Formulae

A *dependency quantified Boolean formula* (DQBF) generalizes a QBF in its allowance for explicit specification of variable dependencies. Syntactically, a DQBF Φ is the same as a QBF except that in Φ_{p} an existential variable y_i is annotated with the set $S_i \subseteq X$ of universal variables referred to by its Skolem function, denoted as $\exists y_{i(S_i)}$, or a universal variable x_j is annotated with the set $H_j \subseteq Y$ of existential variables referred to by its Herbrand function, denoted as $\forall x_{j(H_j)}$, where S_i and H_j are called the *support sets* of y_i and x_j, respectively. However, either the dependencies for the existential variables or the dependencies for the universal variables (but not both) shall be specified. That is, a prenex DQBF is in either of the two forms:

$$\text{S-form:} \qquad \forall x_1 \cdots \forall x_n \exists y_{1(S_1)} \cdots \exists y_{m(S_m)}.\phi, \text{ and} \tag{1}$$

$$\text{H-form:} \qquad \forall x_{1(H_1)} \cdots \forall x_{n(H_n)} \exists y_1 \cdots \exists y_m.\phi, \tag{2}$$

where ϕ is some quantifier-free formula. Note that *the syntactic quantification order in the prefix of a DQBF is immaterial and can be arbitrary* because the variable dependencies are explicitly specified by the support sets. Such quantification with dependency specification corresponds to the Henkin quantifier [16].[1]

By the above syntactic extension of DQBFs, the inputs of the Skolem (respectively Herbrand) function of an existential (respectively universal) variable can be explicitly specified, rather than inferred from the syntactic quantification order. That is, an existential variable y_i (respectively universal variable x_j) can be specified to be semantically independent of a universal variable (respectively an existential variable) whose syntactic scope covers y_i (respectively x_j). Unlike the totally ordered set formed by those of a QBF, the support sets of the existential or universal variables of a DQBF form a partially ordered set in general. This extension makes DQBFs potentially more succinct in expressive

[1] Henkin quantifiers in their original proposal [16] specify dependencies for existential variables only. The dependencies are extended in this paper to universal variables.

power than QBFs [18].

For the semantics, the truth and falsity of a DQBF can be interpreted by the existence of Skolem and Herbrand functions. Precisely an S-form (respectively H-form) DQBF is true (respectively false) if and only if its Skolem (respectively Herbrand) functions exist for the existential (respectively universal) variables while the specified variable dependencies are satisfied. Consequently, Skolem functions serve as the model to a true S-form DQBF whereas Herbrand functions serve as the countermodel to a false H-form DQBF.

Alternatively, the truth and falsity of a DQBF can be understood from a game-theoretic viewpoint. Essentially an S-form DQBF can be interpreted as a game played by one $\forall$-player and m noncooperative $\exists$-players [18]. An S-form DQBF is true if and only if the $\exists$-players have winning strategies, which correspond to the Skolem functions. Similarly an H-form DQBF can be interpreted as a game played by one $\exists$-player and n noncooperative $\forall$-players. An H-form DQBF is false if and only if the $\forall$-players have winning strategies, which correspond to the Herbrand functions.

Example 2. *The S-form DQBF $\Phi = \forall x \exists y_{(x)}.(y \vee x) \wedge (\neg x \vee \neg y)$ is true since its matrix Φ_m becomes a tautology by substituting variable y in Φ_m with its Skolem function $F_y = \neg x$. That is, the existential player has a winning strategy by choosing $y = \neg x$. On the other hand, the H-form DQBF $\Phi = \forall x_{(y)} \exists y.(y \vee x) \wedge (\neg x \vee \neg y)$ is false since its matrix Φ_m becomes unsatisfiable by substituting variable x with its Herbrand function $F_x = y$. That is, the universal player has a winning strategy by choosing $x = y$.*

As was shown in [7, 8], an S-form DQBF Φ can be converted to a *logically equivalent*[2] QBF Φ' by formula expansion on the universal variables. Assume that universal variable x_1 is to be expanded in Formula (1) and $x_1 \notin S_1 \cup \cdots \cup$

[2]That is, Φ and Φ' characterize the same set of Skolem-function models (by properly relating the existential variables of Φ' to those of Φ).

S_{k-1} and $x_1 \in S_k \cap \cdots \cap S_m$. Then Formula (1) can be expanded to

$$\forall x_2 \cdots \forall x_n \exists y_{1(S_1)} \cdots \exists y_{k-1(S_{k-1})}$$
$$\exists y_{k(S_k[x_1/0])} \exists y_{k(S_k[x_1/1])} \cdots \exists y_{m(S_m[x_1/0])} \exists y_{m(S_m[x_1/1])} . \phi|_{x_1=0} \wedge \phi|_{x_1=1},$$

where $S_i[x_1/v]$ denotes x_1 in S_i is substituted with logic value $v \in \{0, 1\}$, and $\phi|_{x_1=v}$ denotes all appearances of x_1 in ϕ are substituted with v including those in the support sets of variables $y_{i(S_i)}$ for $i = k, \ldots, m$. (The subscript of the support set of an existential variable is helpful for tracing expansion paths. Different expansion paths of an existential variable result in distinct existential variables.) Such expansion can be repeatedly applied for every universal variables. The resultant formula after expanding all universal variables is a QBF, whose variables are all existentially quantified. As to be shown in Section 3.2, expansion can be applied also to H-form DQBFs.

3. DQBF Properties

3.1. Negation vs. Complement

In the light of QBF certification, where there always exists either a Skolem-function model or a Herbrand-function countermodel to a QBF, one intriguing question is whether or not the same property carries to DQBFs as well. To answer this question, we distinguish two operators, *negation* (symbolized by "$\neg$") and *complement* (by "$\sim$"), for DQBFs. Let Φ_S and Φ_H be Formulae (1) and (2), respectively. By negation, we define

$$\neg\Phi_S = \exists x_1 \cdots \exists x_n \forall y_{1(S_1)} \cdots \forall y_{m(S_m)} . \neg\phi \text{ and} \tag{3}$$
$$\neg\Phi_H = \exists x_{1(H_1)} \cdots \exists x_{n(H_n)} \forall y_1 \cdots \forall y_m . \neg\phi. \tag{4}$$

By complement, we define

$$\sim\!\Phi_S = \exists x_{1(H'_1)} \cdots \exists x_{n(H'_n)} \forall y_1 \cdots \forall y_m . \neg\phi \text{ and} \tag{5}$$
$$\sim\!\Phi_H = \exists x_1 \cdots \exists x_n \forall y_{1(S'_1)} \cdots \forall y_{m(S'_m)} . \neg\phi, \tag{6}$$

where $H'_i = \{y_j \in Y \mid x_i \notin S_j\}$ and $S'_k = \{x_l \in X \mid y_k \notin H_l\}$, which follow what we call the *complementary principle* of the Skolem and Herbrand support sets.

By the above definitions, one verifies that $\neg\neg\Phi = \Phi$, $\sim\sim\Phi = \Phi$, and $\neg\sim\Phi = \sim\neg\Phi$. Moreover, because the Skolem functions of Φ_S, if they exist, are exactly the Herbrand functions of $\neg\Phi_S$, and the Herbrand functions of Φ_H, if they exist, are exactly the Skolem functions of $\neg\Phi_H$, the following proposition holds.

Proposition 1. *DQBFs under the negation operation obey the* law of the excluded middle*. That is, a DQBF is true if and only if its negation is false.*

Since any DQBF can be converted to a logically equivalent QBF by formula expansion, it also explains that the law of the excluded middle should hold under negation for DQBFs as it holds for QBFs.

A remaining question is whether or not the complement of DQBFs obeys the law of the excluded middle. The answer to this question is in general negative as we show below. Based on the existence of Skolem and Herbrand functions, we classify DQBFs into four categories:

$$\begin{aligned}
\mathcal{C}_S &= \{\Phi \mid \Phi \text{ is true and } {\sim}\Phi \text{ is false}\},\\
\mathcal{C}_H &= \{\Phi \mid \Phi \text{ is false and } {\sim}\Phi \text{ is true}\},\\
\mathcal{C}_{SH} &= \{\Phi \mid \Phi \text{ and } {\sim}\Phi \text{ are true for S-form } \Phi, \text{ or false for H-form } \Phi\}, \text{ and}\\
\mathcal{C}_{\emptyset} &= \{\Phi \mid \Phi \text{ and } {\sim}\Phi \text{ are false for S-form } \Phi, \text{ or true for H-form } \Phi\}.
\end{aligned}$$

Note that if $\Phi \in \mathcal{C}_S$, then ${\sim}\Phi \in \mathcal{C}_H$; if $\Phi \in \mathcal{C}_H$, then ${\sim}\Phi \in \mathcal{C}_S$; if $\Phi \in \mathcal{C}_{SH}$, then ${\sim}\Phi \in \mathcal{C}_{SH}$; if $\Phi \in \mathcal{C}_\emptyset$, then ${\sim}\Phi \in \mathcal{C}_\emptyset$.

Under the above DQBF partition, observe that the complement of DQBFs obeys the law of the excluded middle if and only if $\mathcal{C}_{SH}$ and $\mathcal{C}_\emptyset$ are empty. In fact, as to be shown, for any QBF Φ, $\Phi \notin \mathcal{C}_{SH} \cup \mathcal{C}_\emptyset$. As a consequence, the complement and negation operations for any QBF Φ coincide, and thus $\neg{\sim}\Phi = \Phi$. However, for general DQBFs, $\mathcal{C}_{SH}$ and $\mathcal{C}_\emptyset$ are not empty as the following two examples show.

Example 3. *Consider the DQBF*

$$\Phi = \forall x_1 \forall x_2 \exists y_{1(x_1)} \exists y_{2(x_2)}.((y_1 \oplus x_1) \wedge (y_2 \overline{\oplus} x_2)) \vee ((y_2 \oplus x_2) \wedge (y_1 \overline{\oplus} x_1)),$$

where symbols "$\oplus$" and "$\overline{\oplus}$" stand for Boolean XOR *and* XNOR *operators, respectively. Φ has Skolem functions, e.g., x_1 and $\neg x_2$ for existential variables y_1 and y_2, respectively, and $\neg{\sim}\Phi$ has Herbrand functions, e.g., y_2 and y_1 for universal variables x_1 for x_2, respectively. That is, $\Phi \in \mathcal{C}_{SH}$.*

Example 4. *Consider the DQBF*

$$\Phi = \forall x_1 \forall x_2 \exists y_{1(x_1)} \exists y_{2(x_2)}.(y_1 \vee \neg x_1 \vee x_2) \wedge (y_2 \vee x_1 \vee \neg x_2) \wedge (\neg y_1 \vee \neg y_2 \vee \neg x_1 \vee \neg x_2).$$

It can be verified that Φ has no Skolem functions, and $\neg{\sim}\Phi$ has no Herbrand functions. That is, $\Phi \in \mathcal{C}_\emptyset$. In fact, the absence of Skolem functions in this example can also be established by DQ-resolution to be shown in Example 10 of Section 4.

By these two examples, the following proposition can be concluded.

Proposition 2. *DQBFs under the complement operation do not obey the law of the excluded middle. That is, the truth (respectively falsity) of a DQBF cannot be decided from the falsity (respectively truth) of its complement.*

Nevertheless, if a DQBF $\Phi \notin \mathcal{C}_{SH} \cup \mathcal{C}_\emptyset$, then its truth and falsity can surely be certified by a Skolem-function model and a Herbrand-function countermodel, respectively.[3] That is, excluding those in $\mathcal{C}_{SH} \cup \mathcal{C}_\emptyset$, a DQBF under the complement operation obeys the law of the excluded middle.

A sufficient condition for a DQBF not in $\mathcal{C}_{SH}$ (equivalently, a necessary condition for a DQBF in $\mathcal{C}_{SH}$) is presented in Theorem 3.

Theorem 3. *Let ϕ be a quantifier-free formula over variables $X \cup Y$, let $\Phi_1 = \forall x_1 \cdots \forall x_n \exists y_{1(S_1)} \cdots \exists y_{m(S_m)}.\phi$ and $\Phi_2 = \forall x_{1(H_1)} \cdots \forall x_{n(H_n)} \exists y_1 \cdots \exists y_m.\phi$ with $S_i \subseteq X$ and $H_i = \{y_j \in Y \mid x_i \notin S_j\}$ (namely $\Phi_2 = \neg{\sim}\Phi_1$). Then there exist Skolem functions $\vec{f} = (f_1, \ldots, f_m)$ for Φ_1 and Herbrand functions $\vec{g} =$*

[3] In general a false S-form DQBF has no Herbrand-function countermodel, and a true H-form DQBF has no Skolem-function model.

$(g_1, \ldots, g_n)$ for Φ_2 only if the composite function vector $\vec{g} \circ \vec{f}$ admits no fixed-point, that is, there exists no truth assignment α to variables $\vec{x} = (x_1, \ldots, x_n)$ such that $\alpha = \vec{g}(\vec{f}(\alpha))$.

Proof. Since Φ_1 is true and has Skolem functions $\vec{f}$, formula $\phi[\vec{y}/\vec{f}]$ must be a tautology. On the other hand, since Φ_2 is false and has Herbrand functions $\vec{g}$, formula $\phi[\vec{x}/\vec{g}]$ must be unsatisfiable. Suppose that the fixed-point condition $\alpha = \vec{g}(\vec{f}(\alpha))$ holds under some truth assignment α to $\vec{x}$. Then $\phi[\vec{y}/\vec{f}]|_\alpha = \phi[\vec{x}/\vec{g}]|_\beta$ for $\beta = \vec{f}(\alpha)$ being the truth assignment to $\vec{y}$. It contradicts with the fact that $\phi[\vec{y}/\vec{f}]$ must be a tautology and $\phi[\vec{x}/\vec{g}]$ must be unsatisfiable. ■

The following two corollaries are immediate from Theorem 3.

Corollary 1. *Following the definitions of Φ_1 and Φ_2 in Theorem 3, let $\vec{f} = (f_1, \ldots, f_m)$ be a Skolem-function model to Φ_1. If $\vec{g} = (g_1, \ldots, g_n)$ satisfies $\alpha = \vec{g}(\vec{f}(\alpha))$ for some truth assignment α to variables $\vec{x} = (x_1, \ldots, x_n)$, then $\vec{g}$ cannot be Herbrand functions for Φ_2.*

Corollary 2. *Following the definitions of Φ_1 and Φ_2 in Theorem 3, let $\vec{g} = (g_1, \ldots, g_n)$ be a Herbrand-function countermodel to Φ_2. If $\vec{f} = (f_1, \ldots, f_m)$ satisfies $\alpha = \vec{g}(\vec{f}(\alpha))$ for some truth assignment α to variables $\vec{x} = (x_1, \ldots, x_n)$, then $\vec{f}$ cannot be Skolem functions for Φ_1.*

A brute-force way to show a DQBF formula Φ is in $\mathcal{C}_S$ (respectively $\mathcal{C}_H$) requires pruning all potential Herbrand (respectively Skolem) functions to $\neg{\sim}\Phi$. However since the number of potential Herbrand (respectively Skolem) functions is doubly exponential in the number N of existential (respectively universal) variables, the complexity of such brute-force pruning is $O(2^{2^N})$. Fortunately, for some special cases, Corollary 1 (respectively Corollary 2) and DQ-consensus (respectively DQ-resolution) to be discussed in Section 4 can be helpful to demonstrate the absence of Herbrand (respectively Skolem) functions. Below are two DQBF examples, one in $\mathcal{C}_S$ and the other in $\mathcal{C}_H$.

Example 5. *Consider the DQBF*

$$\Phi = \forall x_1 \forall x_2 \exists y_{1(x_1)} \exists y_{2(x_2)}.(y_1 \overline{\oplus} x_1) \vee (y_2 \overline{\oplus} x_2).$$

It can be verified that Φ has Skolem functions $f_1(x_1) = x_1$ and $f_2(x_2) = 0$ for existential variables y_1 and y_2, respectively. On the other hand, by Corollary 1 we verify that $\neg{\sim}\Phi$ admits no Herbrand functions. Assume $\neg{\sim}\Phi$ has Herbrand functions $g_1(y_2)$ and $g_2(y_1)$ for universal variables x_1 and x_2, respectively. Then we deduce $g_1(f_2) = g_1(0)$ and $g_2(f_1) = g_2(x_1)$. By the fixed-point condition, we have the system of Boolean equations:

$$\begin{aligned} x_1 &= g_1(0), \text{ and} \\ x_2 &= g_2(x_1), \end{aligned}$$

which always has a fixed-point solution $x_1 = g_1(0)$ and $x_2 = g_2(g_1(0))$ independent of the choices of functions g_1 and g_2. It follows that no Herbrand functions can exist for $\neg{\sim}\Phi$. Therefore, $\Phi \in \mathcal{C}_S$.

Example 6. *Consider the DQBF*

$$\Phi = \forall x_1 \forall x_2 \exists y_{1(x_1)} \exists y_{2(x_2)}.(\neg y_1 \vee \neg x_2) \wedge (\neg y_2 \vee \neg x_1) \wedge (y_1 \vee y_2 \vee x_1 \vee x_2).$$

It can be verified that Φ has no Skolem functions, and $\neg{\sim}\Phi$ has Herbrand functions y_2 and y_1 for universal variables x_1 and x_2, respectively. That is, $\Phi \in \mathcal{C}_H$. Notice that the absence of Skolem functions can also be established by DQ-resolution to be discussed in Section 4.

The following corollary shows that $\Phi \notin \mathcal{C}_{SH}$ for any QBF Φ.

Corollary 3. *For any QBF Φ, the Skolem-function model and Herbrand-function countermodel cannot co-exist.*

Proof. If a QBF is false, its Skolem-function model does not exist and the corollary trivially holds. Without loss of generality, assume a true QBF is of the form $\Phi = \exists \vec{y_1} \forall \vec{x_1} \cdots \exists \vec{y_n} \forall \vec{x_n}.\phi$. Let $\{\vec{y_1} = \vec{f_1}(), \ldots, \vec{y_n} = \vec{f_n}(\vec{x_1}, \ldots, \vec{x_{n-1}})\}$ be a model for Φ. Further by contradiction assume there exist a countermodel $\{\vec{x_1} = \vec{g_1}(\vec{y_1}), \ldots, \vec{x_n} = \vec{g_n}(\vec{y_1}, \ldots, \vec{y_n})\}$. So the fixed-point condition is $\{\vec{x_1} = \vec{g_1}(\vec{f_1}()), \ldots, \vec{x_n} = \vec{g_n}(\vec{f_1}(), \ldots, \vec{f_n}(\vec{x_1}, \ldots, \vec{x_{n-1}}))\}$. Since no cyclic dependency presents in the fixed-point equations, the set of equations always has a solution.

HerbrandConstruct
input: a false S-form DQBF $\Phi = \forall x_1 \cdots \forall x_n \exists y_{1(S_1)} \cdots \exists y_{m(S_m)}.\phi$, and the number n of universal variables
output: Herbrand-functions $(g_1, \cdots, g_n)$ of $\neg{\sim}\Phi$
01 $H_n := \{y_i \in Y \mid x_n \notin S_i\}$
02 **if** $(n > 1)$
03 $\quad \Phi_{\mathrm{exp}} := \mathit{FormulaExpand}(\Phi,\, x_n)$;
04 $\quad \vec{g}^{\dagger} := \mathit{HerbrandConstruct}(\Phi_{\mathrm{exp}},\, n-1)$;
05 $\quad$ **if** $(\vec{g}^{\dagger} = \emptyset)$ **return** $\emptyset$;
06 $\quad \vec{g} := \mathit{VariableMerge}(\vec{g}^{\dagger})$;
07 $\quad$ **for each** assignment α to H_n
08 $\quad\quad$ **if** $(\phi[x_1/g_1, \ldots, x_{n-1}/g_{n-1}]|_{\alpha, x_n=0}$ is unsatisfiable)
09 $\quad\quad\quad g_n(\alpha) = 0$;
10 $\quad\quad$ **if** $(\phi[x_1/g_1, \ldots, x_{n-1}/g_{n-1}]|_{\alpha, x_n=1}$ is unsatisfiable)
11 $\quad\quad\quad g_n(\alpha) = 1$;
12 $\quad\quad$ **else return** $\emptyset$;
13 **else**
14 $\quad$ **for each** assignment α to H_n
15 $\quad\quad$ **if** $(\phi|_{\alpha, x_n=0}$ is unsatisfiable)
16 $\quad\quad\quad g_n(\alpha) = 0$;
17 $\quad\quad$ **if** $(\phi|_{\alpha, x_n=1}$ is unsatisfiable)
18 $\quad\quad\quad g_n(\alpha) = 1$;
19 $\quad\quad$ **else return** $\emptyset$;
20 **return** $(g_1, \ldots, g_n)$;
end

Figure 1: Algorithm: Herbrand-function Construction

In other words, due to the complete ordering of the prefix of a QBF, a fixed-point exists. By Theorem 3, the Skolem-function model and Herbrand-function countermodel cannot co-exist. ∎

A sufficient condition for a DQBF not in $\mathcal{C}_\emptyset$ can be characterized by procedure *HerbrandConstruct* as shown in Figure 1. Note that although the algorithm computes Herbrand functions of $\neg{\sim}\Phi_S$ for a false S-form DQBF Φ_S, it can be used to compute Skolem functions of $\neg{\sim}\Phi_H$ for a true H-form DQBF Φ_H by taking as input the negation of the formula.

Given a false S-form DQBF Φ with $n \geq 1$ universal variables, procedure *HerbrandConstruct* in line 1 collects the support set H_n for universal variable x_n. Let $H_n = \{y_{a_1}, \ldots, y_{a_k}\}$ and the rest be $\{y_{a_{k+1}}, \ldots, y_{a_m}\}$. It then recursively constructs the Herbrand functions of the formula expanded on x_n until $n = 1$.

By formula expansion on x_n in line 3, variables $\{y_{a_{k+1}}, \ldots, y_{a_m}\}$, which depend on x_n, are instantiated in Φ_{exp} into two copies, say, $\{y'_{a_{k+1}}, y''_{a_{k+1}}, \ldots, y'_{a_m}, y''_{a_m}\}$. Then the *VariableMerge* step in line 6 lets $g_i = g_i^{\dagger}[y'_{a_{k+1}}/y_{a_{k+1}}, y''_{a_{k+1}}/y_{a_{k+1}}, \ldots, y'_{a_m}/y_{a_m}, y''_{a_m}/y_{a_m}]$. In constructing the Herbrand function g_n of x_n, each assignment α to H_n is examined. Since Herbrand functions aim to falsify ϕ, the value of $g_n(\alpha)$ is set to the x_n value that makes $\phi[x_1/g_1, \ldots, x_{n-1}/g_{n-1}]|_{\alpha}$ unsatisfiable.

The complexity of procedure *HerbrandConstruct* is dominated by formula expansion in line 3, functional substitution in line 6, and function g_n derivation in lines 7-12 and 14-19. The computation is bounded within exponential space and thus double exponential time. From a theoretical viewpoint, it might seem unwise to solve an NEXPTIME problem with a double exponential time algorithm. In practice the complexity bounds may not well reflect the actual performance in real-life applications. For example, most modern satisfiability (SAT) solvers deploy conflict driven clause learning (CDCL) algorithms [4], which may potentially take exponential space resources, for solving an NP-complete problem. Apart from the complexity issue, the main purpose of *HerbrandConstruct* is to theoretically demonstrate that all QBFs and some DQBFs do not belong to $\mathcal{C}_{\emptyset}$ as to be shown.

Note that the implementation of *VariableMerge* in line 6 of procedure *HerbrandConstruct* is not unique. In theory, as long as no violation of variable dependency is incurred, any substitution can be applied. In practice, however the choice of substitution may affect the strength of the algorithm *HerbrandConstruct* in returning (non-empty) Herbrand functions. Note also that the procedure *HerbrandConstruct* is incomplete and may return an empty solution even when Herbrand functions exist. It however can be turned into a complete algorithm if all functional substitutions without variable dependency violation are considered in *VariableMerge*. This modification however may result in formidable computation overhead in practice.

Theorem 4. *Given a false S-form DQBF Φ, algorithm HerbrandConstruct re-*

turns either nothing or correct Herbrand functions, which falsify $\neg{\sim}\Phi$.

Proof. Observe first that the functions returned by the algorithm satisfy the support-set dependencies for the universal variables. It remains to show that $\phi[x_1/g_1, \ldots, x_n/g_n]$ is unsatisfiable. By contradiction, suppose there exists an assignment β to the existential variables Y such that $\phi[x_1/g_1, \ldots, x_n/g_n]|_\beta = 1$. Let $v \in \{0, 1\}$ be the value of $g_n|_\alpha$ for α being the projection of β on $H_n \subseteq Y$. Then $\phi[x_1/g_1, \ldots, x_{n-1}/g_{n-1}, x_n/v]|_\beta = 1$. However it contradicts with the way how $g_n|_\alpha$ is constructed. Hence the returned Herbrand functions $(g_1, \ldots, g_n)$, if they are not empty, are indeed correct Herbrand functions. ■

The following corollary shows that $\Phi \notin \mathcal{C}_\emptyset$ for any QBF Φ.

Corollary 4. *If Φ is a false QBF and its universal variables $x_1, \ldots, x_n$ follow the QBF's prefix order, algorithm HerbrandConstruct always returns non-empty Herbrand functions.*

Proof. We prove the statement by induction on the number of universal variables. For the base case, without loss of generality consider QBF $\Phi = \exists y_1 \cdots \exists y_k$ $\forall x \exists y_{k+1} \cdots \exists y_m.\phi$. After line 1, *HerbrandConstruct* enters line 14. Since the QBF is false and has only one universal variable x, expanding on x yields a purely existentially quantified unsatisfiable formula: $\exists y_1 \cdots \exists y_k(\exists y'_{k+1} \cdots \exists y'_m.\phi|_{x=0} \wedge$ $\exists y''_{k+1} \cdots \exists y''_m.\phi|_{x=1})$. By its unsatisfiability, for every assignment α to $y_1, \cdots, y_k$, formula $\exists y'_{k+1} \cdots \exists y'_m.\phi|_{\alpha,x=0} \wedge \exists y''_{k+1} \cdots \exists y''_m.\phi|_{\alpha,x=1}$ must be unsatisfiable. Since $\exists y'_{k+1} \cdots \exists y'_m.\phi|_{\alpha,x=0}$ and $\exists y''_{k+1} \cdots \exists y''_m.\phi|_{\alpha,x=1}$ share no common variables, at least one of them must be unsatisfiable. Hence the procedure returns a non-empty Herbrand function.

For the inductive step, assume the previous recursive calls for $k = 1, \ldots, n-1$ of *HerbrandConstruct* do not return $\emptyset$. We show that the current call for $k = n$ cannot return $\emptyset$. Expanding Φ on x_n yields $\Phi_{\text{exp}} = \forall x_1 \cdots \forall x_{n-1} \exists y_{1(S_1)} \cdots \exists y_{k(S_k)}$ $(\exists y'_{k+1(S_{k+1})} \cdots \exists y'_{m(S_m)}.\phi|_{x_n=0} \wedge \exists y''_{k+1(S_{k+1})} \cdots \exists y''_{m(S_m)}.\phi|_{x_n=1})$. By the inductive hypothesis, functions $g_1^\dagger, \cdots, g_{n-1}^\dagger$ are returned. Moreover, $g_i^\dagger$ for any $i = 1, \ldots, n-1$ is independent of y'_j and y''_j for $j = k+1, \ldots, m$. So we construct $g_i = g_i^\dagger$. Since $g_1, \ldots, g_{n-1}$ have been constructed in a way such that $\exists y_1 \cdots \exists y_k (\exists y'_{k+1} \cdots \exists y'_m . \phi[x_1/g_1, \cdots, x_{n-1}/g_{n-1}]|_{x_n=0} \wedge \exists y''_{k+1} \cdots \exists y''_m . \phi[x_1/g_1, \cdots, x_{n-1}/g_{n-1}]|_{x_n=1})$ is unsatisfiable, under every assignment α to $y_1, \cdots, y_k$ formula $(\exists y'_{k+1} \cdots \exists y'_m . \phi[x_1/g_1, \cdots, x_{n-1}/g_{n-1}]|_{x_n=0} \wedge \exists y''_{k+1} \cdots \exists y''_m . \phi[x_1/g_1, \cdots, x_{n-1}/g_{n-1}]|_{x_n=1})$ is unsatisfiable. Moreover, since $\exists y'_{k+1} \cdots \exists y'_m .\ \phi[x_1/g_1, \cdots, x_{n-1}/g_{n-1}]|_{x_n=0}$ and $\exists\, y''_{k+1} \cdots \exists y''_m .\ \phi[x_1/g_1, \cdots, x_{n-1}/g_{n-1}]|_{x_n=1}$ do not share any variables, at least one of them must be unsatisfiable. So g_n is returned. ■

Note that the above proof does not explicitly perform the substitution $g_i = g_i^\dagger[y'_{a_{k+1}}/y_{a_{k+1}}, y''_{a_{k+1}}/y_{a_{k+1}}, \ldots, y'_{a_m}/y_{a_m}, y''_{a_m}/y_{a_m}]$ in *VariableMerge* because all g_i in fact do not depend on primed or double-primed variables in the QBF case.

Procedure *HerbrandConstruct* is useful in deriving Herbrand functions not only for QBFs but also for general DQBFs as the following example suggests.

Example 7. *Consider the DQBF*

$$\Phi = \forall x_1 \forall x_2 \exists y_{1(x_1)} \exists y_{2(x_2)} . \phi$$

with

$$\phi = (y_1 \vee x_2) \wedge (y_2 \vee x_1) \wedge (\neg y_1 \vee \neg y_2 \vee \neg x_1 \vee \neg x_2).$$

HerbrandConstruct($\Phi, 2$) computes Herbrand functions for $\neg\sim\Phi$ with the following steps. Expanding Φ on x_2 yields $\Phi_{\text{exp}} = \forall x_1 \exists y_{1(x_1)} \exists y'_2 \exists y''_2 . \phi|_{x_2=0} \wedge \phi|_{x_2=1}$ with $\phi|_{x_2=0} = (y_1) \wedge (y'_2 \vee x_1)$ and $\phi|_{x_2=1} = (y''_2 \vee x_1) \wedge (\neg y_1 \vee \neg y''_2 \vee \neg x_1)$. The recursive call to HerbrandConstruct($\Phi_{\text{exp}}, 1$) determines the value of function $g_1^\dagger(y'_2, y''_2)$ under every assignment α to (y'_2, y''_2). In particular, $g_1^\dagger(0,0) = 0$ due to $\phi_{\text{exp}} = (y_1) \wedge (x_1) \wedge (x_1)$; $g_1^\dagger(0,1) = 0$ (or 1) due to $\phi_{\text{exp}} = (y_1) \wedge (x_1) \wedge (\neg y_1 \vee \neg x_1)$; $g_1^\dagger(1,0) = 0$ due to $\phi_{\text{exp}} = (y_1) \wedge (x_1)$; $g_1^\dagger(1,1) = 1$ due to $\phi_{\text{exp}} = (y_1) \wedge (\neg y_1 \vee \neg x_1)$. So $g_1^\dagger(y'_2, y''_2) = y'_2 y''_2$ (or y''_2), and $g_1(y_2) = g_1^\dagger[y'_2/y_2, y''_2/y_2] = y_2$.

Returning to HerbrandConstruct($\Phi, 2$), we have $\phi[x_1/g_1] = (y_1 \vee x_2) \wedge (y_2) \wedge (\neg y_1 \vee \neg y_2 \vee \neg x_2)$. The value of function g_2 for each assignment α to y_1 can be determined with $g_2(0) = 0$ due to $\phi[x_1/g_1]|_{y_1=0} = (x_2) \wedge (y_2)$ and $g_2(1) = 1$ due *to* $\phi[x_1/g_1]|_{y_1=1} = (y_2) \wedge (\neg y_2 \vee \neg x_2)$*. That is,* $g_2(y_1) = y_1$*. The computed* g_1 *and* g_2 *indeed make* $\phi[x_1/g_1, x_2/g_2] = (y_1) \wedge (y_2) \wedge (\neg y_1 \vee \neg y_2)$ *unsatisfiable.*

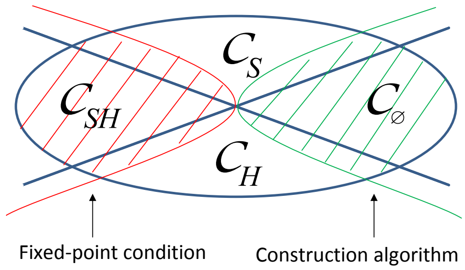

Figure 2: Four DQBF categories and regions characterized by Theorems 3 and 4.

Since the DQBF subset $\mathcal{C}_S \cup \mathcal{C}_H$ obeys the law of the excluded middle under the complement operation, Theorems 3 and 4 provide a tool to test whether a DQBF Φ can be equivalently expressed as $\neg\sim\Phi$, that is, whether a DQBF has either a Skolem-function model or a Herbrand-function countermodel. Figure 2 shows the four DQBF categories and the regions characterized by Theorems 3 and 4.

3.2. Formula Expansion on Existential Variables

Formula expansion on existential variables for DQBFs can be achieved by negation using De Morgan's law and expansion on universal variables. It leads to the following expansion rule, which is dual to expanding universal variables.

Proposition 3. *Given a DQBF* $\forall x_{1(H_1)} \cdots \forall x_{n(H_n)} \exists y_1 \cdots \exists y_m . \phi$*, assume without loss of generality that* y_1 *is to be expanded with* $y_1 \notin H_1 \cup \cdots \cup H_{k-1}$ *and* $y_1 \in H_k \cap \cdots \cap H_n$*. The formula can be expanded to*

$$\forall x_{1(H_1)} \cdots \forall x_{k-1(H_{k-1})} \forall x_{k(H_k[y_1/0])} \forall x_{k(H_k[y_1/1])} \cdots \forall x_{n(H_n[y_1/0])} \forall x_{n(H_n[y_1/1])}$$
$$\exists y_2 \cdots \exists y_m . \phi|_{y_1=0} \vee \phi|_{y_1=1},$$

where $H_i[y_1/v]$ *denotes* y_1 *in* H_i *is substituted with logic value* $v \in \{0,1\}$*, and* $\phi|_{y_1=v}$ *denotes all appearances of* y_1 *in* ϕ *are substituted with* v *including those in the support sets of variables* $x_{i(H_i)}$ *for* $i = k, \ldots, n$*.*

Such expansion can be repeatedly applied for every existential variables. The resultant formula after expanding all existential variables is a QBF. Note that, when Skolem functions are concerned rather than Herbrand functions, the support sets of the existential variables should be listed and can be obtained from H_i by the aforementioned complementary principle.

Example 8. *Consider expanding variable y_1 of DQBF*

$$\Phi = \forall x_{1(y_1)} \forall x_{2(y_2)} \forall x_{3(y_3)} \exists y_1 \exists y_2 \exists y_3 . \phi .$$

By De Morgan's law and expansion on a universal variable, we obtain

$$\begin{aligned}
\neg\neg\Phi &= \neg\exists x_{1(y_1)} \exists x_{2(y_2)} \exists x_{3(y_3)} \forall y_1 \forall y_2 \forall y_3 . \neg\phi \\
&= \neg\exists x_{1(0)} \exists x_{1(1)} \exists x_{2(y_2)} \exists x_{3(y_3)} \forall y_2 \forall y_3 . \neg\phi|_{y_1=0} \wedge \neg\phi|_{y_1=1} \\
&= \forall x_{1(0)} \forall x_{1(1)} \forall x_{2(y_2)} \forall x_{3(y_3)} \exists y_2 \exists y_3 . \phi|_{y_1=0} \vee \phi|_{y_1=1} .
\end{aligned}$$

3.3. Prenex and Non-prenex Conversion

This section studies some syntactic rules that allow localization of quantifiers to sub-formulae. We focus on the truth (namely the Skolem-function model), while similar results can be concluded by duality for the falsity (namely the Herbrand-function countermodel), of a formula.

The following proposition shows the localization of existential quantifiers to the sub-formulas of a disjunction.

Proposition 4. *The DQBF*

$$\forall \vec{x} \exists y_{1(S_1)} \cdots \exists y_{m(S_m)} . \phi_A \vee \phi_B ,$$

where $\forall \vec{x}$ denotes $\forall x_1 \cdots \forall x_n$, sub-formula ϕ_A (respectively ϕ_B) refers to variables $X_A \subseteq X$ and $Y_A \subseteq Y$ (respectively $X_B \subseteq X$ and $Y_B \subseteq Y$), is logically equivalent to

$$\forall \vec{x_c} \left(\forall \vec{x_a} \exists y_{a_1 (S_{a_1} \cap X_A)} \cdots \exists y_{a_p (S_{a_p} \cap X_A)} \phi_A \vee \forall \vec{x_b} \exists y_{b_1 (S_{b_1} \cap X_B)} \cdots \exists y_{b_q (S_{b_q} \cap X_B)} \phi_B \right) ,$$

where variables $\vec{x_c}$ are in $X_A \cap X_B$, variables $\vec{x_a}$ are in $X_A \backslash X_B$, variables $\vec{x_b}$ are in $X_B \backslash X_A$, $y_{a_i} \in Y_A$, and $y_{b_j} \in Y_B$.

Proof. A model to the former expression consists of every truth assignment to X and the induced Skolem function valuation to Y. Since every such combined assignment to $X \cup Y$ either satisfies ϕ_A or ϕ_B, by collecting those satisfying ϕ_A (respectively ϕ_B) and projecting to variables $X_A \cup Y_A$ (respectively $X_B \cup Y_B$) the model (i.e., the Skolem functions for $\vec{y_a}$ and $\vec{y_b}$) to the latter expression can be constructed. (Note that, for a quantifier $\exists y_i$ splitting into two, one for ϕ_A and the other for ϕ_B, in the latter expression, they are considered distinct and have their own Skolem functions.)

In addition, the Skolem functions for $\forall \vec{x_a} \exists y_{a_1 (S_{a_1} \cap X_A)} \cdots \exists y_{a_p (S_{a_p} \cap X_A)} \phi_A|_\alpha$ and those for $\forall \vec{x_b} \exists y_{b_1 (S_{b_1} \cap X_B)} \cdots \exists y_{b_q (S_{b_q} \cap X_B)} \phi_B|_\alpha$ under every assignment α to $\vec{x_c}$ can be collected and combined to form a model for the former expression. In particular the respective Skolem functions $f_{a_j}|_\alpha$ and $f_{b_k}|_\alpha$ under α for y_{a_j} and y_{b_k} originating from the same quantifier y_i in the former expression are merged into one Skolem function $f_i = \bigvee_\alpha \left(\chi_\alpha (f_{a_j}|_\alpha \vee f_{b_k}|_\alpha)\right)$, where χ_α denotes the characteristic function of α, e.g., $\chi_\alpha = x_1 x_2 \neg x_3$ for $\alpha = (x_1 = 1, x_2 = 1, x_3 = 0)$.

■

Example 9. *Consider the QBF*

$$\Phi = \forall x_1 \exists y_1 \forall x_2 \exists y_2 \forall x_3 \exists y_3 . \phi_A \vee \phi_B$$

with ϕ_A refers to variables x_1, x_2, y_1, y_2 and ϕ_B refers to x_2, x_3, y_2, y_3. It has the following equivalent DQBF expressions.

$$\begin{aligned}
\Phi &= \forall x_1 \forall x_2 \forall x_3 \exists y_{1(x_1)} \exists y_{2(x_1,x_2)} \exists y_{3(x_1,x_2,x_3)} . \phi_A \vee \phi_B \\
&= \forall x_1 \forall x_2 \forall x_3 \left(\exists y_{1(x_1)} \exists y_{2(x_1,x_2)} \phi_A \vee \exists y_{2(x_2)} \exists y_{3(x_2,x_3)} \phi_B \right) \\
&= \forall x_2 \left(\forall x_1 \exists y_{1(x_1)} \exists y_{2(x_1,x_2)} \phi_A \vee \forall x_3 \exists y_{2(x_2)} \exists y_{3(x_2,x_3)} \phi_B \right)
\end{aligned}$$

In contrast, conventionally the quantifiers of the QBF can only be localized to

$$\forall x_1 \exists y_1 \forall x_2 \left(\exists y_2 \phi_A \vee \exists y_2 \forall x_3 \exists y_3 \phi_B \right).$$

The following proposition shows the localization of existential quantifiers to a sub-formula of a conjunction.

Proposition 5. *The DQBF*

$$\forall \vec{x} \exists y_{1(S_1)} \cdots \exists y_{k(S_k)} . \phi_A \wedge \phi_B,$$

where $\forall \vec{x}$ *denotes* $\forall x_1 \cdots \forall x_n$*, sub-formula* ϕ_A *(respectively* ϕ_B*) refers to variables* $X_A \subseteq X$ *and* $Y_A \subseteq Y$ *(respectively* $X_B \subseteq X$ *and* $Y_B \subseteq Y$*), is logically equivalent to*

$$\forall \vec{x} \exists y_{2(S_2)} \cdots \exists y_{k(S_k)} . \left(\exists y_{1(S_1 \cap X_A)} \phi_A \right) \wedge \phi_B,$$

for $y_1 \notin Y_B$*.*

Proof. The proposition follows from the fact that the Skolem function of y_1 is purely constrained by ϕ_A only, and is the same for both expressions. Note that the former formula is equivalent to $\forall \vec{x} \exists y_{1(S_1 \cap X_A)} \cdots \exists y_{k(S_k)} . \phi_A \wedge \phi_B$. ■

Essentially DQBFs allow tighter localization of quantifier scopes than QBFs. On the other hand, converting a non-prenex QBF to the prenex form may incur the size increase of support sets of existential variables due to the linear (or complete order) structure of the prefix. With DQBFs, such spurious increase can be eliminated.

4. DQ-Resolution and DQ-Consensus

The rule of Q-resolution for QBFs can be naturally extended to DQBFs as follows. For an S-form DQBF $\Phi = \forall x_1 \cdots \forall x_n \exists y_{1(S_1)} \cdots \exists y_{m(S_m)} . \phi$ in PCNF, a clause $C \in \phi$ is called *minimal* if, for every literal $l \in C$ with $var(l) = x_i \in X$, there exists some $l' \in C$ with $var(l') = y_j \in Y$ such that $x_i \in S_j$. Otherwise, C is *non-minimal.* A non-minimal clause C can be reduced to a minimal clause $C^\dagger$ by removing from C its universal literals

$$L_\forall = \{ l \in C \mid var(l) = x_i \in X \text{ and } x_i \notin S_j \text{ for all } var(l') = y_j \in Y \text{ with } l' \in C \},$$

i.e., $C^\dagger = C \backslash L_\forall$. This reduction process is called the $\forall_D$*-reduction.* Similarly, by duality in an H-form DQBF $\Phi = \forall x_{1(H_1)} \cdots \forall x_{n(H_n)} \exists y_1 \cdots \exists y_m . \phi$ in PDNF, a cube $C \in \phi$ is called *minimal* if, for every literal $l \in C$ with $var(l) = y_i \in Y$, there exists some $l' \in C$ with $var(l') = x_j \in X$ such that $y_i \in H_j$. Otherwise, C is *non-minimal*. A non-minimal cube C can be reduced to a minimal cube $C^\dagger$ by removing from C its existential literals

$$L_\exists = \{l \in C \mid var(l) = y_i \in Y \text{ and } y_i \notin H_j \text{ for all } var(l') = x_j \in X \text{ with } l' \in C\},$$

i.e., $C^\dagger = C \backslash L_\exists$. This reduction process is called the $\exists_D$*-reduction*.

DQ-resolution and *DQ-consensus* of DQBFs are the same as Q-resolution and Q-consensus of QBFs, respectively, except that the $\forall$-reduction and $\exists$-reduction are replaced by $\forall_D$-reduction (for S-form DQBFs) and $\exists_D$-reduction (for H-form DQBFs). The following theorem states the soundness of DQ-resolution.

Theorem 5. *Given an S-form DQBF Φ in PCNF, Φ is false if there exists a DQ-resolution sequence leading to an empty clause.*

Proof. Let ϕ be the matrix of Φ in CNF. Assume $\phi = \phi' \wedge C$ for some non-minimal clause $C = (l_1 \vee \cdots \vee l_n \vee l^*)$ with $var(l^*)$ being a universal variable not in S_j for all existential variables $y_j = var(l_i)$, $i = 1, \ldots, n$. Let Ψ be a DQBF same as Φ expect for the clause C being replaced by $(l_1 \vee \cdots \vee l_n)$. The logical equivalence between Φ and Ψ can be easily established by expanding Φ and Ψ on the universal variable $var(l^*)$. Notice that we only need to expand on universal variables since S-form DQBFs are of concern. By the aforementioned expansion rule, it is easily seen that Φ and Ψ after expansion converge to the same formula. Consequently $\forall_D$-reduction is sound. That is, reducing a non-minimal clause of a DQBF to its minimal form preserves logical equivalence. On the other hand, resolution is sound regardless of the quantification prefix. Since both resolution and $\forall_D$-reduction of DQ-resolution are sound, the derived resolvents and their reduced clauses are logically implied by the original formula. Therefore, as long as an empty clause can be obtained through DQ-resolution, the DQBF must be false. ■

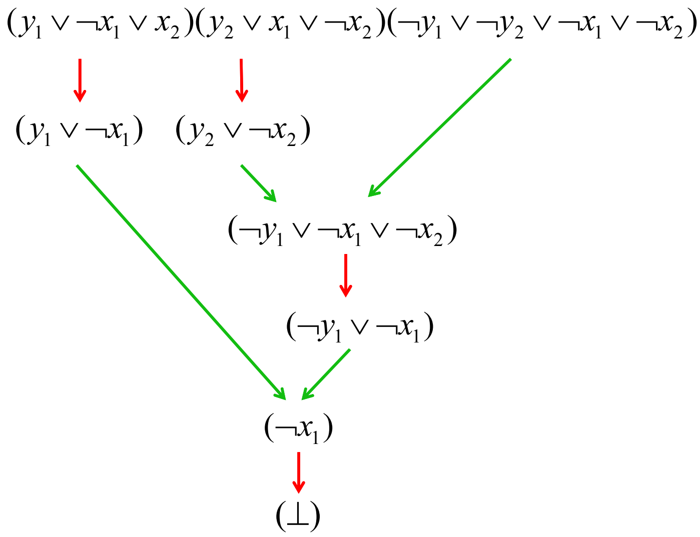

Figure 3: DQ-resolution proof of a false DQBF

Example 10. *Consider the DQBF of Example 4*

$$\forall x_1 \forall x_2 \exists y_{1(x_1)} \exists y_{2(x_2)}.(y_1 \vee \neg x_1 \vee x_2) \wedge (y_2 \vee x_1 \vee \neg x_2) \wedge (\neg y_1 \vee \neg y_2 \vee \neg x_1 \vee \neg x_2),$$

whose falsity is established by the DQ-resolution proof shown in Figure 3, where a clause with a single incoming edge is obtained through $\forall_D$-reduction from its antecedent clause, and a clause with two incoming edges is obtained through resolution from its two antecedent clauses.

Similar to Theorem 5, one can establish by duality the soundness of DQ-consensus.

Theorem 6. *Given an H-form DQBF Φ in PDNF, Φ is true if there exists a DQ-consensus sequence leading to an empty cube.*

PROOF. The proof is similar to that of Theorem 5, and is omitted. ■

Note that the proofs of Theorems 5 and 6 do not carry to H-form DQBFs in PCNF and S-form DQBFs in PDNF, respectively, because expansions on the universal variables of an H-form DQBF and on the existential variables of an S-form DQBF are illy defined.

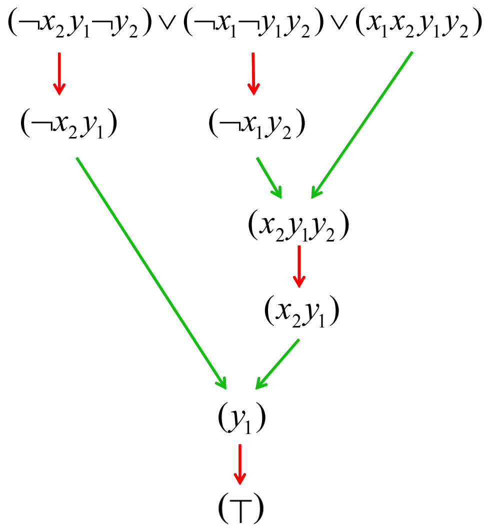

Figure 4: DQ-consensus proof for $\sim\neg\Phi$

On the other hand, let DQ-resolution be similarly defined on H-form DQBFs in PCNF with the following modified rule, denoted $\forall_{\widetilde{D}}$-reduction, by removing from a clause C under the complementary principle its universal literals

$$\{l \in C \mid var(l) = x_i \in X \text{ and } y_j \in H_i \text{ for all } var(l') = y_j \in Y \text{ with } l' \in C\}$$

and DQ-consensus be defined on S-form DQBFs in PDNF with the following modified rule, denoted $\exists_{\widetilde{D}}$-reduction, by removing from a cube C under the complementary principle its existential literals

$$\{l \in C \mid var(l) = y_i \in Y \text{ and } x_j \in S_i \text{ for all } var(l') = x_j \in X \text{ with } l' \in C\}.$$

Then DQ-resolution and DQ-consensus under these modified reduction rules are unsound in general as the following example shows.

Example 11. *Consider the S-form DQBF*

$$\Phi = \forall x_1 \forall x_2 \exists y_{1(x_1)} \exists y_{2(x_2)}.(\neg x_2 \wedge y_1 \wedge \neg y_2) \vee (\neg x_1 \wedge \neg y_1 \wedge y_2) \vee (x_1 \wedge x_2 \wedge y_1 \wedge y_2).$$

As can be verified, it is false due to the absence of Skolem function models. However, $\exists_{\widetilde{D}}$-reduction may lead to a DQ-consensus proof of an empty cube as *shown in Figure 4, where a cube with a single incoming edge is obtained through $\exists_{\tilde{D}}$-reduction from its antecedent cube, and a cube with two incoming edges is obtained through consensus from its two antecedent cubes. The consensus proof in turn asserts that the H-form $\sim\neg\Phi$ is true. Therefore $\exists_{\tilde{D}}$-reduction is unsound for S-form DQBFs. (Similarly $\forall_{\tilde{D}}$-reduction is unsound for H-form DQBFs.)*

Although $\forall_{\tilde{D}}$ and $\exists_{\tilde{D}}$ reductions are unsound for general DQBFs, they are sound for DQBFs in the category of $\mathcal{C}_S \cup \mathcal{C}_H$ as shown in the following propositions.

Proposition 6. *The $\exists_{\tilde{D}}$-reduction rule is sound for DQ-consensus on an S-form DQBF Φ in PDNF provided that $\Phi \in \mathcal{C}_S \cup \mathcal{C}_H$.*

Proof. Given an S-form DQBF $\Phi \in \mathcal{C}_S \cup \mathcal{C}_H$ in PDNF, let Π be a DQ-consensus proof with the extended $\exists_{\tilde{D}}$-reduction leading to an empty cube. Suppose Π' be the same as Π except that every cube of Π is negated to a clause by De Morgan's laws. Then it can be verified that, for the (S-form) DQBF $\sim\Phi$ (in PCNF), Π' corresponds to a DQ-resolution proof with the standard $\forall_D$-reduction leading to an empty clause. By Theorem 5, since $\sim\Phi$ is an S-form DQBF with a DQ-resolution proof leading to an empty clause, $\sim\Phi$ is false. Finally, since $\Phi \in \mathcal{C}_S \cup \mathcal{C}_H$, we conclude Φ must be true. ∎

By duality, one can similarly establish the following proposition.

Proposition 7. *The $\forall_{\tilde{D}}$-reduction rule is sound for DQ-resolution on an H-form DQBF Φ in PCNF provided that $\Phi \in \mathcal{C}_S \cup \mathcal{C}_H$.*

By Theorem 5, Theorem 6, Proposition 6, and Proposition 7, we know that all the reduction rules $\forall_D$, $\forall_{\tilde{D}}$, $\exists_D$, and $\exists_{\tilde{D}}$ can be safely applied for DQBFs in $\mathcal{C}_S \cup \mathcal{C}_H$, as in the special case of QBFs.

Although DQ-resolution and DQ-consensus are sound as shown in Theorems 5 and 6, they are unfortunately incomplete in proving the truth and falsity of DQBFs as we show below.

Theorem 7. *DQ-resolution is incomplete in proving the truth and falsity of S-form DQBFs.*

Proof. The theorem can be established by the following DQBF.

$$\Phi = \forall x_1 \forall x_2 \exists y_{1(x_1)} \exists y_{2(x_2)}.(y_1 \vee y_2 \vee x_1) \wedge (\neg y_1 \vee \neg y_2 \vee x_1) \wedge (y_1 \vee y_2 \vee \neg x_1 \vee \neg x_2) \wedge (\neg y_1 \vee y_2 \vee \neg x_1 \vee x_2) \wedge (y_1 \vee \neg y_2 \vee \neg x_1 \vee x_2) \wedge (\neg y_1 \vee \neg y_2 \vee \neg x_1 \vee \neg x_2).$$

It can be verified that Φ is false (i.e., no Skolem function models), and yet no DQ-resolution steps can be made. ■

Similarly one can establish the following claim.

Theorem 8. *DQ-consensus is incomplete in proving the truth and falsity of H-form DQBFs.*

Since the truth of an S-form (respectively H-form) DQBF is defined by the existence of Skolem (respectively Herbrand) functions, the definition is not consistent with the existence of DQ-consensus (respectively DQ-resolution) proofs. For the QBF case, a Q-resolution (respectively Q-consensus) proof can be both an evidence for the absence of Skolem (respectively Herbrand) functions and an evidence for the existence of Herbrand (respectively Skolem) functions [5, 6]. For the DQBF case, unfortunately there is no such nice property. For an S-form (respectively H-form) DQBF Φ, a DQ-resolution (respectively DQ-consensus) proof can only certify the absence of Skolem (respectively Herbrand) functions, thus soundly proving the formula is false (respectively true), but cannot guarantee the existence of Herbrand (respectively Skolem) functions for the H-form (respectively S-form) DQBF $\sim\neg\Phi$.

Even though DQ-resolution and DQ-consensus are incomplete, they are useful, due to their soundness, in DQBF evaluation since resolvent clauses and consensus cubes can be used as learnt clauses for S-form DQBFs and learnt cubes for H-form DQBFs, respectively. In fact, one important point to study the complement operator, in contrast to the negation operator, is to see whether the modified $\exists_{\widetilde{D}}$-reduction holds for cube learning in the evaluation of S-form DQBFs in PCNF, similar to $\exists$-reduction for the cube learning in PCNF QBF evaluation [13], as affirmatively answered by Proposition 7 within the DQBF category of $\mathcal{C}_S \cup \mathcal{C}_H$.

Moreover, by Propositions 6 and 7, if for some true S-form (respectively false H-form) DQBF Φ we can prove $\Phi = \sim\neg\Phi$ (namely if $\Phi \in C_S$ or $\Phi \in C_H$), then we can soundly use a DQ-consensus (respectively DQ-resolution) proof for the H-form (respectively S-form) DQBF $\sim\neg\Phi$ as the evidence of the existence of Skolem (respectively Herbrand) functions for Φ.

On the other hand, in light of Theorem 2 of QBFs, one might hope that similar algorithms exist for DQBFs in converting DQ-resolution proofs to Herbrand functions and converting DQ-consensus proofs to Skolem functions. However it is, in general, impossible due to the non-emptiness of $\mathcal{C}_{SH}$ and $\mathcal{C}_{\emptyset}$. Nevertheless, the possibility that such algorithms exist for DQBFs in $\mathcal{C}_S \cup \mathcal{C}_H$ is not ruled out, and our result may provide insight for the development.

5. Applications

DQBF evaluation is a new field with potential broad applications. Its development is underway. To date there is only one search based DQBF solver [12] extended from the Q-DPLL algorithm [4, 10]. We note that the framework provided by the QBF solver sKizzo [2], which is based on Skolemization, can also be naturally extended to DQBF solving. In addition to evaluation, certification of DQBFs, the focus of this work, is equally important in enabling practical applications.

One of the potential applications of DQBFs is topologically constrained logic synthesis [21], where a set of unknown components in a given Boolean network is to be synthesized such that the resultant network behavior conforms to a system specification. Figure 5 depicts one such example, where the network consists of four known and four unknown function components each with two inputs. Given the fixed connection of the network topology and some Boolean relation specifying the set of allowed input-output values, the Boolean functions of the four unknown components are to be synthesized.

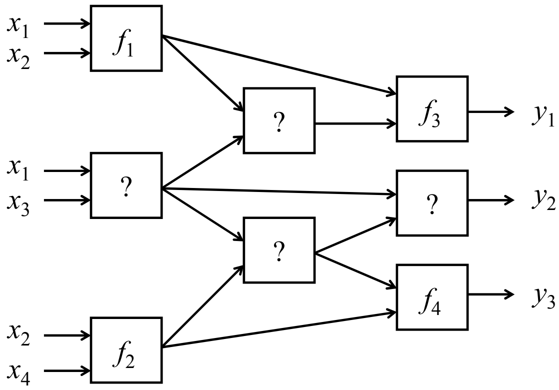

Figure 5: A network of known and unknown logic components

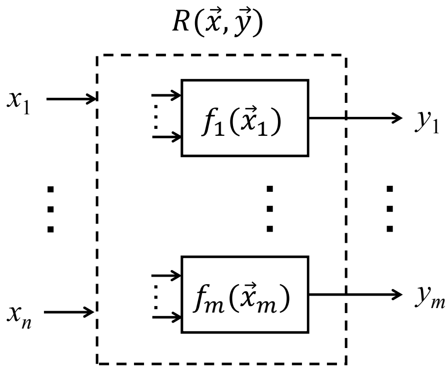

Figure 6: Function derivation from a Boolean relation

A special problem of topologically constrained logic synthesis is shown in Figure 6, where m unknown functions $f_1, \ldots, f_m$ are to be synthesized from a Boolean relation specification $R(\vec{x}, \vec{y})$ with $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_m)$ such that $y_i = f_i(\vec{x}_i)$ with $\vec{x}_i$, a sub-vector of $\vec{x}$, being the pre-specified input constraint of f_i, for $i = 1, \ldots, m$. It can be naturally expressed by the S-form DQBF

$$\forall x_1 \cdots \forall x_n \exists y_{1(\vec{x}_1)} \cdots \exists y_{m(\vec{x}_m)}.R(\vec{x}, \vec{y}). \tag{7}$$

Then the Skolem functions to the DQBF correspond to the desired synthesis solution. This synthesis problem is trivially the same as deriving Herbrand functions to the H-form DQBF

$$\exists x_1 \cdots \exists x_n \forall y_{1(\vec{x}_1)} \cdots \forall y_{m(\vec{x}_m)}.\neg R(\vec{x}, \vec{y}). \tag{8}$$

The above problem is an extension of the (input-unconstrained) Boolean relation determinization problem considered in [5, 6, 14]. Notice that the QBF equivalents to Formulae (7) and (8) may suffer from formula explosion due to the enforcement of independencies among variables by the aforementioned formula expansion.

6. Prior Work

IF logic [17] with the game-theoretical semantics is known to violate the law of the excluded middle. A simple example is the IF logic formula $\forall x \exists y_{/x}.(x = y)$ for $x, y \in \{0, 1\}$, where $y_{/x}$ indicates the independence of y on x [11]. It assumes that not only y is independent of x, but also is x independent of y. That is, it is equivalent to $\forall x_{()} \exists y_{()}.(x = y)$ in our dependency notation. In a game-theoretic viewpoint, neither the $\exists$-player nor the $\forall$-player has a winning strategy. Therefore this formula is neither true nor false, and has no equivalent DQBF since any DQBF can always be expanded into a QBF, whose truth and falsity can be fully determined.

On the other hand, the game-theoretical semantics of IF logic, when extended to DQBFs, does not provide a fully meaningful approach to synthesizing Skolem and Herbrand functions. Unlike the unimportance of the syntactic quantification order in our formulation, the semantic game of IF logic should be played with respect to the prefix order. Since different orders correspond to different games, the semantics is not directly useful in our considered synthesis application.

Henkin quantifiers in their original form [16] specified only the dependencies of existential variables on universal variables. Such restricted dependencies were assumed in early IF logic [17] research. As was argued in [11], the dependency of universal variables on existential variables are necessary to accomplish a symmetric treatment on the falsity, in addition to truth, of an IF logic formula.

With such extension, IF logic formulae can be closed under negation. However, how the dependencies of existential variables and universal variables relate to each other was not studied. The essential notion of Herbrand functions was missing. In contrast, our formulation on DQBFs treats Skolem and Herbrand functions on an equal footing. Unlike [11], we restrict a formula to be of either S-form or H-form, rather than simultaneous specification of dependencies for existential and universal variables. This restriction makes the synthesis of Skolem and Herbrand functions for DQBFs more natural.

Prior work [18, 8] assumed DQBFs are of S-form only. In [18], a DQBF was formulated as a game played by a ∀-player and multiple noncooperative ∃-players. This game formulation is fundamentally different from that of IF-logic. The winning strategies, if they exist, of the ∃-players correspond to the Skolem functions of the DQBF. This game interpretation can be naturally extended to H-form DQBFs.

The soundness of DQ-resolution was briefly mentioned in [12]. In contrast, we formalized DQ-resolution and DQ-consensus for S-form and H-form DQBFs, respectively, and studied their soundness and completeness issues.

7. Conclusion

The syntax and semantics of DQBFs presented in this paper made DQBFs a natural extension of QBFs from a certification viewpoint. Basic DQBF properties, including formula negation, complement, expansion, prenex and non-prenex form conversion, and resolution, were shown. Our formulation is adequate for applications where Skolem/Herbrand functions are of concern.

It remains open whether there exists more elaborated resolution and consensus rules that are both sound and complete for DQBF evaluation. Also the precise characterization of DQBFs in $\mathcal{C}_S \cup \mathcal{C}_H$ remains to be established, and algorithms that convert resolution and consensus proofs to Herbrand and Skolem functions, respectively, for DQBFs in $\mathcal{C}_S \cup \mathcal{C}_H$ remain to be obtained.

Acknowledgments

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